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Characterisation of symmetries of unlabelled triangulations

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Abstract

We derive a constructive decomposition of unlabelled triangulations by giving a full characterisation of their symmetries. This result will enable us to deduce a complete enumerative description of unlabelled cubic planar graphs.

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1 Introduction

1.1 Related work and motivation

The enumeration of graphs embeddable on a surface, in particular planar graphs, has been among the most studied problems in enumerative combinatorics. The breakthrough result by McDiarmid, Steger, and Welsh [9] states that if $P(n)$ denotes the number of labelled planar graphs on n vertices, then $(P(n)/n!)^{1/n}$ converges to a limit γ as $n \to \infty$, which is called the *growth constant* of labelled planar graphs. An upper bound on γ was obtained by Osthus, Prömel, and Taraz [10], using triangulations and probabilistic methods. Bender, Gao, and Wormald [1] showed that the number of labelled 2-connected planar graphs on n vertices is asymptotically $26.1^{n+o(n)}n!$, giving a lower bound for $P(n)$, using the singularity analysis of generating functions arising from the decomposition of planar graphs along connectivity. Further analysing the singularities and singular types of the generating functions in [1], Giménez and Noy [6] proved that $P(n) \sim c\, n^{-7/2}\, 27.2^n\, n!$ for an analytic computable constant $c > 0$.

Despite extensive studies on graphs on a surface and related structures, many structural and enumerative problems concerning *unlabelled* (i.e. non-isomorphic) graphs on a surface are still open. While some subfamilies of unlabelled planar graphs such as outerplanar graphs [2] and series parallel graphs [5] have been enumerated, the fundamental problem of the asymptotic number of *all* unlabelled planar graphs is still open.

Richmond and Wormald [11] proved a conjecture by Tutte [12] that almost all *planar maps* (i.e. graphs that are *embedded* on the sphere) are asymmetric. In other words, almost all planar maps have no non-trivial automorphisms. The situation for *planar graphs* (i.e. graphs that are *embeddable* on the sphere) is however very different: McDiarmid, Steger, and Welsh [9] showed that almost all planar graphs have exponentially many automorphisms. This implies that one cannot deduce the asymptotic number of unlabelled planar graphs from that of labelled planar graphs.

One of the most important concepts used in the enumeration of graphs is *constructive decompositions* of graphs. Such constructive decompositions can be interpreted as functional operations of generating functions which encode the enumerative information of graphs. A seminal example of constructive decompositions is Tutte's decomposition [12]: every 2-connected graph is characterised by three disjoint subclasses of graphs, each of which is decomposed into smaller building blocks, where 3-connected graphs form a base case, and the building blocks construct all possible 2-connected graphs. Chapuy et al. [4] used this decomposition to construct a grammar that allows to transfer enumeration results for 3-connected planar graphs to planar graphs. Since 3-connected planar graphs have a unique embedding up to orientation, enumerating 3-connected labelled *maps* gives rise to an enumeration of labelled planar graphs.

For unlabelled graphs, however, the situation is more difficult: due to symmetries of the graphs and maps considered, the constructive decomposition does not yield a 1-1 correspondence. In order to deal with unlabelled graphs, it is therefore necessary to gain a better understanding of the symmetries of planar graphs.

1.2 Main results and techniques

In this work we derive a complete description of the automorphisms of unlabelled planar triangulations. Because the *dual* of triangulations are 3-connected *cubic* planar graphs, these characterisations of symmetries of unlabelled triangulations can be interpreted in terms of symmetries of unlabelled 3-connected cubic planar graphs. Using the enumeration grammar obtained in [4], we can transfer these symmetry information to those of unlabelled cubic planar graphs, leading to the complete description of unlabelled cubic planar graphs [8]. We believe that the insight gained in this work can be applied to study the symmetry and component structure of unlabelled planar graphs, in particular that of 3-connected unlabelled planar graphs, by carefully characterising planar graphs with different types of symmetries.

Symmetries of triangulations can be of two different types: *reflective* or *rotative*. We develop a constructive decomposition that depends on whether the triangulation T has reflective symmetries, rotative symmetries, or both. In either case, T contains a unique subgraph G from a class of base cases, which we call *girdles*, *spindles*, and *skeletons*. A *girdle* is essentially a cycle with a few additional vertices and edges that is invariant under the reflection. It divides T into two *sides* that are isomorphic. A *spindle* is a system of paths—again with some additional vertices and edges—between the two unique invariant vertices, edges, or faces. It divides T into several near-triangulations which we call the *segments* of the spindle. If T has a rotation of order k, then the set of all segments decomposes into sets of size k consisting of isomorphic near-triangulations. Finally, a *skeleton* is the union of all girdles that arise from the reflective symmetries of T. Similar to the spindle, the skeleton divides T into *segments*, and those segments can be partitioned into equally sized sets of isomorphic segments.

Using these structures, we obtain our main result.

Theorem 1.1 *Triangulations with reflective, rotative, or both symmetries have the following constructive decomposition:*

(i) The set of triangulations with reflective symmetries is obtained from the set of girdles and the set of near-triangulations that are sides of girdles;

(ii) the set of triangulations with rotative symmetries is obtained from the set of spindles and the set of near-triangulations that are segments of spindles; and

(iii) the set of triangulations with both reflective and rotative symmetries is obtained from the set of skeletons and the set of near-triangulations that are segments of skeletons.

2 Preliminaries and notation

All graphs and triangulations in the following are *unlabelled* and do not contain loops or double edges. All maps are considered to be maps on the sphere. We refer to the vertices, edges, and faces of a triangulation as its *cells* of dimension 0, 1, and 2, respectively. Two cells of different dimension are called *incident* if one is contained in the (topological) boundary of the other. For every cell c of a given dimension d the numbers of incident cells of dimensions $d+1$ (mod 3) and $d+2$ (mod 3) coincide. We call this number the *degree* of c and denote it by $d(c)$. The set of cells incident with a given cell c has a cyclic order $(c_1, c_2, \ldots, c_{2d(c)})$ in which two cells are consecutive if and only if they are incident in the triangulation. This order is unique up to orientation. Two cells c_α, c_β with $\alpha, \beta \in \{1, 2, \ldots, 2d(c)\}$ are said to *lie opposite* at c if $|\alpha - \beta| = d(c)$.

A cell c of a triangulation T is *invariant* under a given automorphism φ of T if $\varphi(c) = c$. In enumerative combinatorics, triangulations are often considered with a given *rooting*; in other words, a certain cell is required to be invariant under all automorphisms that are considered. All triangulations in this paper will have a root c_0, which might be a vertex, an edge, or a face. We denote by $\mathrm{Aut}(c_0, T)$ the group of automorphisms of T under which c_0 is invariant.

A basic building block of our constructions will be *near-triangulations*. A near-triangulation N is a plane graph in which one face is marked to be the *outer face*; this face is bounded by a cycle of any length while all other faces, the *inner faces*, are bounded by triangles. The vertices and edges on the boundary of the outer face are the *outer vertices* and *outer edges* of N, all other vertices and edges are *inner vertices* and *inner edges*.

In order to describe automorphisms in $\mathrm{Aut}(c_0, T)$, it is enough to describe their action on the cells incident with c_0. Indeed, if $\varphi, \psi \in \mathrm{Aut}(c_0, T)$ satisfy $\varphi(c) = \psi(c)$ for all cells c incident with c_0, then a simple induction shows that $\varphi = \psi$. On the other hand, the possible actions on the cells incident with c_0 can easily be described using the cyclic sequence of those cells described above: if a nontrivial automorphism $\varphi \in \mathrm{Aut}(c_0, T)$ changes the orientation of the sequence, we call it *reflective*; otherwise, we call it *rotative*.

3 Constructive decomposition of triangulations

3.1 Reflective symmetries

If $\varphi \in \mathrm{Aut}(c_0, T)$ is reflective, then precisely two of the cells incident with c_0 are invariant under φ. A recursive construction by Tutte [13] shows that there is a cyclic sequence $(c_0, \dots, c_k)$ of cells such that

(i) each c_i is invariant under φ;

(ii) two cells c_i, c_j are incident if and only if $|i - j| = 1$; and

(iii) for each i, the cells c_{i-1} and c_{i+1} lie opposite at c_i.

The subgraph G of T consisting of all vertices and edges that either lie in the above sequence or on the boundary of a face in the sequence is the *girdle* of T with respect to φ (see Figure 1(a)). The cells $c_0, \dots, c_k$ are the *central cells* of G, the other vertices and edges of G are its *outer cells*. The girdle divides T into two *sides* that are isomorphic near-triangulations, thus decomposing T into its girdle and two isomorphic near-triangulations.

It is important to note that not all near-triangulations can occur as a side of some triangulation T with reflective symmetry. Indeed, since our triangulations do not have double edges, a side of T can not have an inner edge connecting two central vertices of the girdle, as such an edge would appear in both sides of T and would thus be part of a double edge.

3.2 Rotative symmetries

If $\mathrm{Aut}(c_0, T)$ contains a rotative symmetry φ, of order m say, then there is a unique cell $c_1 \neq c_0$ that is also invariant under φ. We call c_0 the *north pole* and c_1 the *south pole* of T. Tutte [13] used this fact to construct a subgraph of T that consists of a shortest path P between the poles and all images $\varphi^i(P)$ of P. This subgraph divides T into m isomorphic near-triangulations. However, this is not a constructive decomposition as the path P is not unique.

We modify Tutte's construction as follows so as to obtain a constructive decomposition. First, we show that there is a sequence $L_1, \dots, L_k$ of graphs in T, called *levels*, having the following properties.

(i) Each L_i is invariant under φ;

(ii) all vertices in L_1 have an edge to c_0;

(iii) all vertices in L_i, $i > 1$, have an edge to L_{i-1}; and

(iv) L_k contains c_1 or its incident vertices.

Then we grow paths $P_1, \dots, P_{d(c_0)}$ from c_0 to c_1 simultaneously. We start each P_i with an edge from c_0 to L_1. Then for each path, we go along L_1 in clockwise direction until we reach a vertex v_i that has a neighbour in L_2. Then we proceed along the leftmost edge from v_i to L_2 and repeat this construction until we reach the last level L_k (see Figure 1(b)).

The spindle divides T into several near-triangulations, the *segments* of T. Since the order of φ is m, the set of segments is partitioned into sets of m isomorphic near-triangulations. Due to the construction of the spindle, there are several restrictions on the distribution of inner edges in a segment.

3.3 Reflective and rotative symmetries

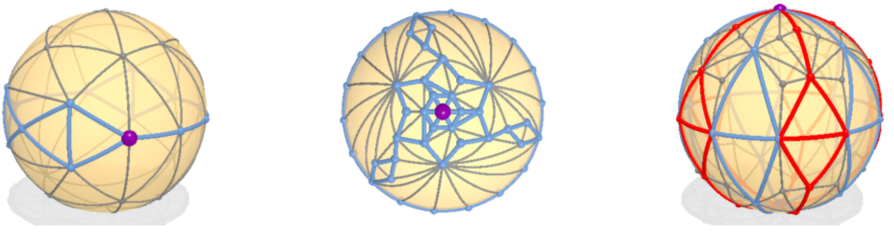
If $\mathrm{Aut}(c_0, T)$ contains both reflective and rotative symmetries, then T contains a girdle for every reflection, say m girdles. Using the fact that the composition of two reflections is a rotation, we can prove that

(i) each girdle contains both poles of T as central cells and

(ii) no other cell is central in more than one girdle.

Note that (ii) is no restriction on the *outer* vertices and edges of girdles. In other words, two girdles can only *intersect* in the poles, but they can *touch* also in other places. By (i), the poles split each girdle into two parts which we call *meridians*. The order in which the meridians are arranged around the north pole c_0 defines a cyclic order $M_1, \dots, M_{2m}$ on the set of meridians. In this order, the meridians M_i and M_{i+m} are the two parts of a girdle for every i. If we apply the reflection corresponding to this girdle, we map M_{i-1} to M_{i+1}. Thus, the meridians with even index are isomorphic, and so are the meridians with odd index.

The union of all meridians is the *skeleton* of T (see Figure 1(c)). Similar to the girdle and spindle, the skeleton divides T into several near-triangulations, the *segments* of T. Applying all automorphisms in $\mathrm{Aut}(c_0, T)$, we see that the set of segments is partitioned into sets of $2m$ isomorphic near-triangulations. Similarly to the case of reflective symmetry, some inner edges are forbidden in segments, since they would result in double edges.



(a) (b) (c)

Fig. 1. Rooted (purple) triangulations with (a) reflective symmetry and its girdle (blue), (b) rotative symmetry and its spindle (blue), (c) both types of symmetries and its skeleton (red/blue). In (c), every two meridians of the same colour are isomorphic.

4 Discussion

Constructive decompositions for *labelled* graphs or maps can be interpreted as functional operations of *generating functions* which encode the enumerative information of these graphs or maps. As for *unlabelled* graphs and maps, the use of *cycle index sums* is more appropriate, since they contain additional information about the automorphisms of these graphs or maps.

Cycle index sums that describe girdles, spindles, or skeletons can be obtained directly from the definitions of these graphs. The near-triangulations that appear in the different cases of the decomposition, on the other hand, can be decomposed in such a way as to obtain functional operations that define their cycle index sums. Using the enumeration grammar from [4], we can transfer the symmetry information encoded in the cycle index sums to those of unlabelled cubic planar graphs, leading to the complete description of unlabelled cubic planar graphs [8]. This will be done in a similar manner as for *labelled* cubic planar graphs [3].

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