Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Enumeration of cubic multigraphs on orientable surfaces

M. Kang, M. Moßhammer, P. Sprüssel [1,2]

Institute of Optimization and Discrete Mathematics
Graz University of Technology
8010 Graz, Austria

W. Fang [3]

LIAFA, UMR CNRS 7089
Université Paris-Diderot
Paris Cedex 13, France

Abstract

Let $\mathbb{S}_g$ be the orientable surface of genus g. We show that the number of edge-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $m = 3k$ edges is asymptotically $d_g \gamma^{-m} m^{5/2(g-1)-1} m!$, where $\gamma^{-1} = \frac{\sqrt{79}}{3} 2^{-1/3}$ and c_g is a constant only dependent on the genus.

Keywords: Cubic graph, Surface, Higher genus, Enumeration

[1] Supported by Austrian Science Fund (FWF): P27290, W1230

[2] Email:{kang,mosshammer,spruessel}@math.tugraz.at

[3] Email:wfang@liafa.univ-paris-diderot.fr

1 Introduction

1.1 Motivation

Since the seminal work of Tutte [14] graphs *embedded* on a surface, called *maps*, have become widely studied. Starting by the number of planar maps computed by Tutte [14], other classes of maps have been extensively studied; for example the number of *rooted* maps on surfaces has been determined by Bender, Canfield, and Richmond [1], and the number of triangulations on arbitrary surfaces by Gao [9,8,7].

On the other hand, analogous problems for graphs that are *embeddable* on a surface are still wide open. The first breakthrough result in this area is due to McDiarmid, Steger, and Welsh [12] who first showed that the number $\mathrm{pl}(n)$ of labelled planar graphs on n vertices has an exponential growth constant, that is, $(\mathrm{pl}(n)/n!)^{1/n}$ converges to a real number γ. Giménez and Noy [10] determined the value γ through the singularity analysis of generating functions arising from a constructive decomposition of planar graphs. They also obtained limit laws for planar graphs with n vertices and $m = \mu n$ edges with $\mu \in (1, 3)$. In view of the classical Erdős-Rényi random graph the more interesting regime is $\mu < 1$, in particular $\mu \sim \frac{1}{2}$: a random graph with edge density μ undergoes the so-called *phase transition* when $\mu \sim \frac{1}{2}$. Kang and Łuczak [11] showed that random *planar* graphs feature an analogous phase transition at $\mu \sim \frac{1}{2}$ and derived the asymptotic number of planar graphs for the regime $\mu < 1$. The main ingredient in their proof is a constructive decomposition that reduces the problem of counting planar graphs to counting *cubic planar multigraphs*.

For graphs embeddable on a surface of positive genus g, Chapuy et al. [3] determined the asymptotic number of such graphs with n vertices and μn edges for the range $\mu \in (1, 3)$ and showed limit laws for this range similar to the planar case. Again, $\mu < 1$ is the more interesting regime from the point of view of the evolution of random graphs and this regime $0 < \mu < 1$ has not been studied.

1.2 Main results and techniques

In this work we derive the asymptotic number of cubic multigraphs embeddable on the orientable surface $\mathbb{S}_g$.

Theorem 1.1 *Let $G_g(m)$ denote the number of cubic multigraphs embeddable* *on the orientable surface $\mathbb{S}_g$ of positive genus g. Then for m divisible by 3*

$$G_g(m) \sim d_g \left(\frac{2^{1/3}3}{\sqrt{79}}\right)^{-m} m^{5/2(g-1)-1}m!,$$

where d_g is a constant depending only on g.

This enumeration result is the essential part in the enumeration of graphs embeddable on $\mathbb{S}_g$ for the regime $\mu < 1$ [5]. Similar to the planar case, the enumeration of graphs embeddable on $\mathbb{S}_g$ can be reduced to that of cubic multigraphs embeddable on $\mathbb{S}_g$ using a constructive decomposition. This decomposition is via *cores* and *kernels*. The core of a graph is the maximal subgraph with minimum degree at least 2 and the kernel is the multigraph of minimum degree at least 3 that has the core as a subdivision. When $\mu \sim \frac{1}{2}$, typical kernels are cubic multigraphs. The reverse direction of this construction enables us to use the number of cubic multigraphs embeddable on $\mathbb{S}_g$ in order to enumerate the graphs embeddable on $\mathbb{S}_g$ for the regime $\mu < 1$ [5].

The first step in our proof is based on a decomposition of multigraphs along connectivity, a construction that reduces the problem of counting cubic multigraphs to counting cubic 3-connected graphs. In the planar case, 3-connected graphs have a unique embedding and thus the enumeration of such graphs is equivalent to the enumeration of the corresponding maps. This relation between 3-connected graphs and maps does not hold for higher genus; similar to Chapuy et al. [3], we instead use a classical result of Robertson and Vitray [13] that guarantees the uniqueness of the embedding if the *facewidth* is large enough. By showing that graphs with small facewidth do not contribute to the leading term in the asymptotic number of 3-connected cubic graphs embeddable on $\mathbb{S}_g$, we can use the aforementioned result of Robertson and Vitray and restrict our attention to 3-connected cubic *maps* on $\mathbb{S}_g$. The dual of such maps are triangulations in which some loops and double edges are forbidden due to the 3-connectivity of the primal. We obtain the number of those triangulations by relating them to simple triangulations as counted by Gao [8]. The main tools in this step are *surgeries*that reduce the genus of the surface.

2 Preliminaries and Notation

By g we will always denote the genus of the orientable surface $\mathbb{S}_g$. A *connected* (multi-)graph is called *embeddable* on $\mathbb{S}_g$, if it has an embedding in $\mathbb{S}_g$ such that every face is homeomorphic to a disc. A (multi-)graph is called embeddable on $\mathbb{S}_g$, if each of its connected components is embeddable on a surface $\mathbb{S}_{g_i}$ and all the genuses sum up to g. Furthermore all (multi-)graphs will be cubic and edge-labelled, unless stated otherwise. In order to obtain the desired asymptotic number of cubic multigraphs, we use generating functions and singularity analysis. Our generating functions will always be exponential generating functions $F(y) = \sum \frac{f_m}{m!} y^m$, where y marks edges and f_m denotes the number of (multi-)graphs in the class F that have m edges. Our aim is to derive a system of relations between various generating functions and to apply singularity analysis in order to obtain the desired asymptotics. To this end, we will use the following notations. We write $F(y) \sim G(y)$ if the coefficients of F and G satisfy $\frac{f_m}{g_m} \to 1$ as $m \to \infty$; $F(y) \precsim G(y)$ if $f_m < g_m$ for all m and $\frac{f_m}{g_m} \to 1$ as $m \to \infty$; and $F(y) \ll G(y)$ if $\frac{f_m}{g_m} \to 0$ as $m \to \infty$.

As the uniqueness of the embedding of a 3-connected planar graph on the sphere does not extend to surfaces of higher genus, we need an additional concept, the *facewidth* of graphs and maps. The facewidth of a map is the minimum number of intersections that a noncontractible circle has with the map. The facewidth of a graph is the maximal facewidth of all its embeddings.

A classical result of Robertson and Vitray [13] says that 3-connected graphs embeddable on $\mathbb{S}_g$ have a unique embedding up to orientation if their facewith is at least $2g+3$. Therefore we can enumerate 3-connected cubic graphs embeddable on $\mathbb{S}_g$ whose facewidth is at least $2g+3$ by counting the corresponding *maps*.

3 Deriving a constructive decomposition

3.1 Triangulations and 3-connected graphs

The duals of 3-connected maps on $\mathbb{S}_g$ are triangulations. The 3-connectivity of the primal translates to the exclusion of some loops and double edges. Denote the class of these triangulations by M_g. In order to count M_g, we develop dominance relations between M_g and triangulations counted by Gao [8], especially with the number of simple triangulations S_g (i.e. triangulations without loops and double edges). A series of *surgeries* on $\mathbb{S}_g$ yield dominance relations between S_g, M_g, and some other intermediate classes of triangulations (see [4] for details).

In this context a *surgery* is the operation of cutting the surface along a loop or double edge and thereby either separating the surface into two surfaces of smaller genus or decreasing the genus of the surface. By adding or removing edges around the holes obtained by surgery we obtain triangulations. This construction leads to dominance relations between triangulations before and after the surgery. These dominance relations result in the asymptotic number of triangulations in M_g.

The final step is to show that the number of all 3-connected cubic maps with a fixed facewidth smaller that $2g+3$ is asymptotically negligible compared to the asymptotic number of 3-connected cubic maps with facewidth at least $2g+3$. Thereby we derive the asymptotic number of 3-connected cubic graphs embeddable on $\mathbb{S}_g$.

3.2 Decomposing along connectivity

The constructive decomposition that allows us to obtain the number of cubic multigraphs embeddable on $\mathbb{S}_g$ from the number of 3-connected cubic graphs is based on the following theorem by Robertson and Vitray [13]: for $k \in \{2, 3\}$, every $(k-1)$-connected graphs G embedded on $\mathbb{S}_g$ with facewidth at least k has a unique k-connected component having the same genus and facewidth as G, while all other k-connected components are planar.

By this theorem for $k = 3$, we can construct every 2-connected cubic multigraph embeddable on $\mathbb{S}_g$ with facewidth at least 3 by first taking the 3-connected component provided by the theorem and then replacing edges (u, v) by a *network*—a 2-connected planar cubic multigraph—and an edge from this multigraph to each of u and v. This results in a *dominance* (but not equality) relation between the corresponding generating functions, because for *planar* graphs, the component provided by the theorem of Robertson and Vitray might depend on the embedding on $\mathbb{S}_g$. However, we can derive an asymptotic formula for 2-connected cubic multigraphs embeddable on $\mathbb{S}_g$, $g > 0$, because almost all such graphs are nonplanar and thus have a unique 3-connected component independent of the embedding.

Analogously, the theorem of Robertson and Vitray for $k = 2$ allows us to construct every connected cubic multigraph embeddable on $\mathbb{S}_g$ with facewidth at least 2 by starting with a 2-connected cubic multigraph with the same genus and facewidth and adding connected cubic planar multigraphs at some of its edges. Again this yields a dominance relation between the corresponding generating functions, but we obtain an asymptotic formula by a similar argument as above.

The connection between connected and general cubic multigraphs embeddable on $\mathbb{S}_g$ is a set construction. A general multigraph contains any number of planar components and at most g connected components with positive genus.

The final step is to show that the number of multigraphs with small facewidth is asymptotically negligible. By cutting the surface along a carefully adjusted circle that witnesses the facewidth we obtain a dominance relation for multigraphs with small facewidth, which in turn shows that the number of these multigraphs is small.

3.3 Generating functions

The constructions above result in the following equations and dominance relations for generating functions. We will denote facewidth conditions using superscripts of the corresponding functions. From the decomposition relating 3-connected graphs and triangulations, we obtain

$$4y\frac{\mathrm{d}}{\mathrm{d}y}T_g^{fw\geq 3}(y) \;\sim\; 4y\frac{\mathrm{d}}{\mathrm{d}y}T_g^{fw\geq 2g+3}(y) \;\sim\; M_g(y) \;\sim\; S_g(y),$$

where $T_g(y)$ counts 3-connected graphs, $M_g(y)$ counts their duals, and $S_g(y)$ counts simple triangulations on $\mathbb{S}_g$. From the decomposition of 2-connected multigraphs we get

$$\begin{aligned} B_g^{fw\geq 3}(y) &\precsim T_g^{fw\geq 3}\left(y + yN(y)\right), \\ B_g^{fw=2}(y) &\ll B_g^{fw\geq 3}(y), \\ N(y) &= \frac{1}{2}S_0\left(y + yN(y)\right), \end{aligned}$$

where $B_g(y)$ counts 2-connected cubic multigraphs and $N(y)$ counts networks. The decomposition of connected cubic multigraphs yields

$$\begin{aligned} C_g^{fw\geq 2}(y) &\precsim B_g^{fw\geq 2}\left(\frac{y}{1-Q(y)}\right), \\ C_g^{fw=1}(y) &\ll C_g^{fw\geq 2}(y), \end{aligned}$$

where $C_g(y)$ counts connected cubic multigraphs and $Q(y)$ counts the planar multigraphs used in the decomposition and is given by the implicit equations

$$\begin{aligned} Q(y) &= \frac{Q^2(y)}{2} + \frac{y^3 A}{2}, \\ 0 &= 4 - 52A^2 + 336\,A^3y^3 + 240A^4 + 1848A^5y^3 + A^6(-448 + 3017y^6) \\ &\qquad -2400A^7y^3 + A^8(256 + 1024y^6) + 4096A^9y^9. \end{aligned}$$

The decomposition of general cubic multigraphs yields

$$G_g(y) \;=\; e^{C_0(y)} \sum_{\sum g_i = g;\, g_i > 0} \frac{1}{k!}\prod_{i=1}^{k} C_{g_i}(y).$$

Using standard methods of singularity analysis developed by Flajolet and Odlyzko [6] we conclude

$$
\begin{aligned}
[y^n]\,T_g(y) &\sim a_g \gamma_T^{-m} m^{5/2(g-1)-1},\\
[y^n]\,B_g(y) &\sim b_g \gamma_B^{-m} m^{5/2(g-1)-1},\\
[y^n]\,C_g(y) &\sim c_g \gamma_C^{-m} m^{5/2(g-1)-1},\\
[y^n]\,G_g(y) &\sim d_g \gamma_C^{-m} m^{5/2(g-1)-1},
\end{aligned}
$$

where a_g, b_g, c_g, and d_g are computable constants depending only on g and $\gamma_T = \frac{3}{8}2^{1/3}$, $\gamma_B = \frac{6}{17}2^{1/3}$, and $\gamma_C = \frac{3}{\sqrt{79}}2^{1/3}$ are constants independent of g. If we are marking vertices instead of edges the growth constants can easily be obtained replacing $3m$ by $2n$. Note that the growth constants do not coincide with the growth constants obtained in [2] as we are allowing multiple edges and loops.

4 Further work

The Kernel-Core method described earlier allows a constructive decomposition of all labelled graphs embeddable on $\mathbb{S}_g$ with n vertices and $m = \mu n$ edges, where $\mu < 1$. This decomposition enables us to not only determine the number of those graphs, but also obtain more insight into their structural properties [5], e.g. the size of the largest component.

References

[1] Bender, E. A., E. R. Canfield and L. B. Richmond, *The asymptotic number of rooted maps on a surface.: II. Enumeration by vertices and faces*, J. Combin. Theory Ser. A **63** (1993), pp. 318–329.
URL
http://www.sciencedirect.com/science/article/pii/009731659390063E

[2] Bodirsky, M., M. Kang, M. Löffler and C. McDiarmid, *Random cubic planar graphs*, Random Struct. Alg. **30** (2007), pp. 78–94.
URL http://dx.doi.org/10.1002/rsa.20149

[3] Chapuy, G., É. Fusy, O. Giménez, B. Mohar and M. Noy, *Asymptotic enumeration and limit laws for graphs of fixed genus*, J. Combin. Theory Ser. A **118** (2011), pp. 748–777.
URL http://dx.doi.org/10.1016/j.jcta.2010.11.014

[4] Fang, W., M. Kang, M.Moßhammer and P. Sprüssel, *Enumeration of cubic multigraphs on orientable surfaces* In preparation.

[5] Fang, W., M. Kang, M.Moßhammer and P. Sprüssel, *Sparse graphs embeddable on orientable surfaces* In preparation.

[6] Flajolet, P. and A. M. Odlyzko, *Singularity analysis of generating functions.*, SIAM J. Discrete Math. **3** (1990), pp. 216–240.
URL
http://dblp.uni-trier.de/db/journals/siamdm/siamdm3.html#FlajoletO90

[7] Gao, Z., *The number of rooted 2-connected triangular maps on the projective plane*, J. Combin. Theory Ser. B **53** (1991), pp. 130–142.
URL http://dx.doi.org/10.1016/0095-8956(91)90058-R

[8] Gao, Z., *The asymptotic number of rooted 2-connected triangular maps on a surface*, J. Combin. Theory Ser. B **54** (1992), pp. 102–112.
URL http://dx.doi.org/10.1016/0095-8956(92)90068-9

[9] Gao, Z., *A pattern for the asymptotic number of rooted maps on surfaces*, J. Combin. Theory Ser. A **64** (1993), pp. 246–264.
URL http://dx.doi.org/10.1016/0097-3165(93)90097-R

[10] Giménez, O. and M. Noy, *Asymptotic enumeration and limit laws of planar graphs*, J. Amer. Math. Soc. **22** (2009), pp. 309–329.
URL http://dx.doi.org/10.1090/S0894-0347-08-00624-3

[11] Kang, M. and T. Łuczak, *Two critical periods in the evolution of random graphs*, Trans. Am. Math. Soc. **364** (2012), pp. 4239–4265.

[12] McDiarmid, C., A. Steger and D. J. A. Welsh, *Random planar graphs*, J. Comb. Theory Ser. B **93** (2005), pp. 187–205.
URL http://dx.doi.org/10.1016/j.jctb.2004.09.007

[13] Robertson, N. and R. Vitray, *Representativity of surface embeddings*, in: B. Korte, L. Lovsz, H. Promel and A. Schrijver, editors, *Algorithms and Combinatorics*, Springer, 1990 pp. 293–328.

[14] Tutte, W. T., *A census of planar maps*, Canad. J. Math. **15** (1963), pp. 249–271.