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On Spanning Structures in Random Hypergraphs

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Abstract

In this note we adapt a general result of Riordan [*Spanning subgraphs of random graphs*, Combinatorics, Probability & Computing **9** (2000), no. 2, 125–148] from random graphs to random r-uniform hypergaphs. We also discuss several spanning structures such as cube-hypergraphs, lattices, spheres and Hamilton cycles in hypergraphs.

Keywords: Random hypergraphs, spanning structures, threshold functions.

1 Introduction

Finding spanning subgraphs is a well studied problem in random graph theory, see e.g. the following monographs on random graphs [4,14]. In the case of hypergraphs not so much is known and it is natural to study the corresponding problems for hypergraphs.

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An r-uniform hypergraph H is a tuple (V, E), where V is its vertex set and $E \subseteq \binom{V}{r}$ the set of edges in H. Further we write $\deg(v)$ for the degree of a vertex v in H: $\deg(v) := |\{e\colon v \in e\}|$, and $\Delta(H)$ denotes the maximum vertex degree in H, i.e. $\Delta(H) := \max_{v\in V} \deg(v)$. We will consider two models of r-uniform random hypergraphs $\mathcal{H}^{(r)}(n,p)$ and $\mathcal{H}^{(r)}(n,m)$. Formally, $\mathcal{H}^{(r)}(n,p)$ is the probability space of all labelled r-uniform hypergraphs with the vertex set $[n]$ where each edge $e \in \binom{[n]}{r}$ is chosen independently of all the other edges with probability p. Similarly one defines $\mathcal{H}^{(r)}(n,m)$ as the probability space of all labelled r-uniform hypergraphs with the vertex set $[n]$ and exactly m edges and considers a uniform measure.

We will shortly write $\mathcal{H}$ for a random graph in one of these classes and all probabilities are with respect to the corresponding model. For $r = 2$ these are the standard models $G(n,p)$ and $G(n,m)$ respectively.

Let $H = H^{(i)}$ be a sequence of fixed r-uniform hypergraphs with n vertices, where $n = n(i) \to \infty$. Then we say that $\mathcal{H}$ contains the graph H asymptotically almost surely (a.a.s.) if the probability that $H^{(i)} \subseteq \mathcal{H}$ tends to 1 as n tends to infinity (here $\mathcal{H} = \mathcal{H}^{(r)}(n,p)$ or $\mathcal{H} = \mathcal{H}^{(r)}(n,m)$). We say that $\hat{p}$ is a threshold function if $\mathbb{P}[H \subseteq \mathcal{H}^{(r)}(n,p)] \to 0$ for $p \ll \hat{p}$ and $\mathbb{P}[H \subseteq \mathcal{H}^{(r)}(n,p)] \to 1$ for $p \gg \hat{p}$ as n tends to infinity. Similarly one defines a threshold function $\hat{m} = \hat{m}(n)$ in the model $\mathcal{H}^{(r)}(n,m)$. It was shown by Bollobás and Thomason [5] that all nontrivial monotone properties have a threshold function. Since subgraph containment is a monotone property it is natural to study the threshold functions for appearance of various structures in random graphs and hypergraphs.

The case of *fixed* (hyper-)graphs was solved by Erdős and Rényi [10](balanced case) and by Bollobás [3]. First spanning structures considered in graphs were perfect matchings [11] and Hamilton cycles [4,17,23]. More recently, the thresholds for the appearance of (bounded degree) spanning trees were studied as well, for the currently best bounds see Montgomery [19,20].

Alon and Füredi [2] studied the question when the random graph $G(n,p)$ contains a given graph G of bounded maximum degree Δ hereby proving the bound $p \geq C(\ln n/n)^{1/\Delta}$ for some absolute constant $C > 0$. In [24] Riordan proved quite a general theorem applicable to various graphs including hypercubes and lattices. Finding thresholds for factors of graphs and hypergraphs was long an open problem where breakthrough was achieved by Johansson, Kahn and Vu [15]. Kahn and Kalai [16] have a general conjecture about the thresholds for the appearance of a given structure (which roughly states that the threshold p with $\mathbb{P}(G \subseteq G(n,p)) = 1/2$ for containment of G is within a factor of $O(\ln n)$ from p_E at which the expected number of copies of G in

$G(n, p_E)$ is 1, where p_E is the so-called expectation threshold).

When one turns to hypergraphs, so apart from perfect matchings and general factors [15], the only other spanning structures that were studied more recently are *Hamilton cycles*. One defines an ℓ-overlapping Hamilton cycle as an r-uniform hypergraph with $n/(r-\ell)$ edges such that for some cyclic ordering of $[n]$ and an ordering of the edges, every edge e_i consists of r consecutive vertices and for any two consecutive edges e_i and e_{i+1} it holds $|e_i \cap e_{i+1}| = \ell$. We say that a hypergraph is ℓ-hamiltonian if it contains an ℓ-overlapping Hamilton cycle (this always requires that $r-\ell$ divides n). Frieze [13] determined the threshold for the appearance of 1-overlapping 3-uniform Hamilton cycle to be $\Theta(\ln n/n^2)$ (when $4|n$) and Dudek and Frieze [7] extended the result to higher uniformities $(2(r-1)|n)$. The divisibilty requirement was improved to the optimal one $((r-1)|n)$ by Dudek, Frieze, Loh and Speiss [9], see also Ferber [12]. More recently, the threshold for the so-called *Berge* Hamilton cycles was studied by Poole [22]. Moreover, Dudek and Frieze [8] determined thresholds for general ℓ-overlapping Hamilton cycles and a randomized algorithm to find $(r-1)$-overlapping Hamilton cycles was given in [1]. For the table of the known thresholds we refer the reader to [8], but generally $\omega(n^{\ell-r})$ is an asymptotically optimal threshold for ℓ-Hamiltonicity (for $\ell \geq 2$ and in most situations even more precise results are known), where $\omega(f)$ is any function g such that $g(n)/f(n) \to \infty$ as $n \to \infty$.

The purpose of this note is to observe that Riordan's proof can be adapted to r-uniform hypergaphs leading to a general theorem about spanning structures in random hypergraphs. We will recover results of Dudek and Frieze [8] on ℓ-hamiltonicity and also discuss thresholds for other spanning structures such as hypergraph hypercubes, hyperlattices and spheres.

Let H be an r-uniform hypergraph with n vertices, one defines $e_H(v) = \max\{e(F) : F \subseteq H, |F| = v\}$ and

$$\gamma(H) = \max_{r+1 \leq v \leq n} \left\{ \frac{e_H(v)}{v-2} \right\}.$$

The following theorem in the case of $r = 2$ was proved by Riordan [24]. However the same conclusion applies to general r-uniform hypergraphs.

Theorem 1.1 *Let $r \geq 2$ be an integer and $H = H^{(i)}$ be a sequence of r-uniform hypergraphs with $n = n(i)$ vertices and $e(H) = \alpha\binom{n}{r} = \alpha(n)\binom{n}{r}$ edges. Let $p = p(n) \colon \mathbb{N} \to [0,1]$ and $p\binom{n}{r}$ be an integer. If the following conditions* *are satisfied*

$$\alpha\binom{n}{r} > \frac{n}{r}, \quad p\binom{n}{r} \to \infty, \quad (1-p)\sqrt{n} \to \infty, \tag{1}$$

and

$$np^{\gamma(H)}\Delta^{-4} \to \infty, \tag{2}$$

then a.a.s. the random r-uniform hypergraphs $\mathcal{H}^{(r)}(n,p)$ and $\mathcal{H}(n,p\binom{n}{r})$ contain a copy of H.

For the proof we refer the reader to the full version of our paper [21].

We immediately obtain the following two corollaries. We state them only in the model $\mathcal{H}^{(r)}(n,p)$, but the corresponding statements follow immediately for $\mathcal{H}^{(r)}(n,m)$ with $m = \lceil p\binom{n}{r}\rceil$ by a standard argument, see [4,24].

Corollary 1.2 *Let r, $\Delta \geq 2$ be integers and $H = H^{(i)}$ a sequence of r-uniform hypergraphs with $n = n(i)$ vertices, $\Delta(H) \leq \Delta$, $e(H) > n/r$ and $\gamma(H) = e(H)/(n-2)$. Then for $p = \omega\left(n^{-1/\gamma(H)}\right)$ the random graph $\mathcal{H}^{(r)}(n,p)$ contains a copy of H a.a.s., while for every $\varepsilon > 0$ we have for $p \leq (1-\varepsilon)(e/n)^{1/\gamma}$ that $\mathbb{P}(H \subseteq \mathcal{H}^{(r)}(n,p)) \to 0$.*

Proof. Since $\gamma(H) \leq (1+o(1))\Delta$ and by monotonicity of the graph containment property we may assume that $p = o(1)$ and thus the conditions (1) and (2) are satisfied. Since Δ is fixed, we obtain from Theorem 1.1 the first part of the claim.

Let X be the number of copies of H in $\mathcal{H}^{(r)}(n,p)$ and we estimate its expectation $\mathbb{E}(X)$ as follows:

$$\mathbb{E}(X) \leq n!p^{e(H)} \leq 3\sqrt{n}(1-\varepsilon)^{e(H)}(n/e)^2 = o(1).$$

Now Markov's inequality $\mathbb{P}(X \geq 1) \leq \mathbb{E}(X)$ yields the second part of the corollary. □

We call a hypergraph H d-regular if every vertex of H has degree d.

Corollary 1.3 *Let $r \geq 2$ be an integer and $H = H^{(i)}$ be a sequence of Δ-regular r-uniform hypergraphs where $\Delta = \omega(\log(n)^{1-1/r})$ but $\Delta = o(n^{1/4})$. Then for every $\varepsilon > 0$ we have that $\mathcal{H}^{(r)}(n,p)$ contains a.a.s. H if $p = (1+\varepsilon)n^{-r/\Delta}$. Furthermore $\mathbb{P}(H \subseteq \mathcal{H}^{(r)}(n,p)) \to 0$ for $p \leq n^{-r/\Delta}$, i.e. $p = n^{-r/\Delta}$ is a sharp threshold for the appearance of copies of H in $\mathcal{H}^{(r)}(n,p)$.*

Proof. Let X count the copies of H in $\mathcal{H}^{(r)}(n,p)$ and for $p \le n^{-r/\Delta}$ we have

$$\mathbb{P}(X \ge 1) \le \mathbb{E}(X) \le n! n^{-re(H)/\Delta} = n! n^{-n} = o(1).$$

Next we bound $\gamma(H)$ as follows: $\Delta/r \le \gamma(H) \le \frac{\Delta}{r}\frac{(\Delta^{1/(r-1)}+1)}{(\Delta^{1/(r-1)}-1)}$. This is obtained from the estimate $e_H(v) \le \min\{\Delta v/r, \binom{v}{r}\}$ by considering two cases whether $v \le \Delta^{1/(r-1)} + 1$ or not. Let $\varepsilon \in (0,1)$ and notice that (1) is satisfied. It also holds that

$$n\left((1+\varepsilon)n^{-r/\Delta}\right)^{\gamma(H)} \Delta^{-4} \ge \left((1+\varepsilon)n^{1/\gamma(H)-r/\Delta}\Delta^{-4r(1+o(1))/\Delta}\right)^{\gamma(H)} \ge \\ \left((1+\varepsilon)n^{-2r/(\Delta^{1+1/(r-1)})}(1+o(1))\right)^{\gamma(H)} \to \infty,$$

and therefore Theorem 1.1 is applicable and the statement follows. □

2 Discussion of Theorem 1.1

Riordan's argument for random graphs in [24] is an elegant second moment argument, where the variance of the random variable X that counts the number of copies of H is estimated carefully by considering the contributions coming from various possible intersections of any two copies of H in the complete r-uniform hypergraph. The overall proof strategy of Theorem 1.1 is the same as in [24] (one works in the model $\mathcal{H}^{(r)}(n,m)$ and then conditions on the number of edges in $\mathcal{H}^{(r)}(n,p)$ to obtain a corresponding result for $\mathcal{H}^{(r)}(n,p)$). In fact, most of the proof can be read along the lines of the original argument apart from Lemmas 4.3 and 4.5 in [24] (and the auxiliary lemmas from [24] essentially state the same but for hypergraphs). However, some complications arise and in particular one needs to generalize Lemma 4.3 from [24] to hypergraphs which requires more case distinctions. Moreover, Lemma 4.5 [24] can be generalized in that one replaces every edge in an r-uniform hypergraph with the clique K_r and proves the lemma by counting trees in the shadow instead of counting hypertrees.

Thus Theorem 1.1 (Corollaries 1.2 and 1.3) states that under some technical conditions the threshold for the appearance of the spanning structure comes from the expectation threshold defined in the introduction. Further it should be noted that the appearance of 1-overlapping Hamilton cycles and also perfect matchings and of general F-factors the structure in question appears as soon as some local obstruction (isolated vertices, no vertices in some copy of a fixed graph F) disappears. Thus, there seem to be two types of behaviour that are responsible for the threshold for the appearance of a bounded degree spanning structure.

3 Applications

In the following we derive asymptotically optimal thresholds for the appearance of various spanning structures in $\mathcal{H}^{(r)}(n,p)$ which are consequences of the Corollaries 1.2 and 1.3.

3.1 Hamilton Cycles

The following is a slightly weaker version of Dudek and Frieze [8].

Corollary 3.1 *For all integers $r > \ell \geq 2$, $(r-\ell)|n$ and $p = \omega(n^{\ell-r})$ the random hypergraph $\mathcal{H}^{(r)}(n,p)$ is ℓ-hamiltonian a.a.s.*

Proof. Denote by $C^{(r,\ell)}$ an ℓ-overlapping Hamilton cycle on n vertices. It is not difficult to see that $\gamma(C^{(r,\ell)}) = \frac{n}{(r-\ell)(n-2)}$. Indeed, let $V \subseteq [n]$ be a set of size $v < n$. Then $C^{(r,\ell)}[V]$ is a union of vertex-disjoint ℓ-overlapping paths, where an ℓ-overlapping path of length s consists of $s(r-\ell)+\ell$ ordered vertices and edges are consecutive segments intersecting in ℓ vertices. This gives: $e(C^{(r,\ell)}[V]) \leq (v-\ell)/(r-\ell)$ and from $\frac{v-\ell}{(r-\ell)(v-2)} \leq \frac{n}{(r-\ell)(n-2)}$ we get $\gamma(C^{(r,\ell)}) = \frac{n}{(r-\ell)(n-2)}$.

Since $e(C^{(r,\ell)}) > n/r$, $\Delta(C^{(r,\ell)}) = \lceil \frac{r}{r-\ell} \rceil$ and $n^{2(r-\ell)/n} \to 1$, Corollary 1.2 implies the statement. □

3.2 Cube-hypergraphs

The r-uniform d-dimensional cube-hypergraph $Q^{(r)}(d)$ was studied in [6] and its vertex set is $V := [r]^d$ and its hyperedges are r-sets of the vertex set V that all differ in one coordinate. Thus, $Q^{(r)}(d)$ has r^d vertices, dr^{d-1} edges and is d-regular. In the case $r = 2$ this is the usual definition of the (graph) hypercube. The following corollary is a direct consequence of Corollary 1.3.

Corollary 3.2 *For all integers $r \geq 2$, $\varepsilon > 0$ and $p = r^{-r} + \varepsilon$ it holds $\mathbb{P}(Q^{(r)}(d) \subseteq \mathcal{H}(r^d, p))$ tends to 1 as d tends to infinity. On the other hand, $\mathbb{P}(Q^{(r)}(d) \subseteq \mathcal{H}(r^d, r^{-r})) \to 0$ as $d \to \infty$.*

We remark that in the case $r = 2$ Riordan [24] proved even better dependence of ε on d, and similar dependence can be shown for $r > 2$.

3.3 Lattices

Another example considered in [24] was the graph of the lattice L_k, whose vertex set is $[k]^2$ and two vertices are adjacent if their Euclidean distance is one. There it is shown that $p = n^{-1/2}$ is asymptotically the threshold. One can view L_k as the cubes $Q^{(2)}(2)$ (these are cycles C_4) glued 'along' the edges. We define the ℓ-overlapping hyperlattice $L^{(r)}(\ell, k)$ as the r-uniform hypergraph where we glue together $(k-1)^2$ copies of $Q^{(r)}(2)$ that overlap on ℓ hyperedges accordingly. Thus, $L^{(2)}(1, k)$ is just the usual graph lattice L_k.

Corollary 3.3 *Let $r \geq 2$ and k be an integer. For $p = \omega\left(n^{-1/2}\right)$ (where $n = (k-2+r)^2$) the random r-uniform hypergraph $\mathcal{H}^{(r)}(n, p)$ contains a copy of $L^{(r)}(r-1, k)$ a.a.s. Moreover, for $p = n^{-1/2}$, $\mathbb{P}(L^{(r)}(r-1, k) \subseteq \mathcal{H}^{(r)}(n, p)) \to 0$ as k (and thus n) tends to infinity.*

Proof. Observe that $L := L^{(r)}(r-1, k)$ has $(k-2+r)^2$ vertices (which can be associated with $[k-2+r]^2$) and $2(k-1)(k-2+r)$ edges.

We aim to show that $e_L(v) \leq 2(v-r)$ for all $v \geq r+1$. We argue similarly as in [24]. Observe that $e_L(v) \leq 2$ for $v = r+1$. Let now L' be an arbitrary subhypergraph of L on $v+1 \leq (k-2+r)^2$ vertices such that $e(L') = e_L(v+1)$. It is easy to see that there is a vertex of degree 2 in L' (take (i, j) such that $(i+1, j)$, $(i, j+1) \notin V(L')$). It follows that then $e_L(v+1) \leq e_L(v) + 2$ for $v > r+1$ giving $e_L(v) \leq 2(v-r)$ for all $v \geq r+1$.

It follows that $\gamma(L) \leq 2$ and applying Theorem 1.1 with $np^\gamma = \omega(1)$ yields the first part. Markov's inequality yields the second part. □

3.4 Spheres

Let $r \geq 3$ and let G be a planar graph on n vertices with a drawing all of whose faces are cycles of length r. We define a sphere S_n^r as an r-uniform hypergraph all of whose edges correspond to the faces of that particular drawing (note that a sphere is not unique). Observe that we get from Euler's formula for planar graphs the condition $2v(S_n^r) - 4 = (r-2)e(S_n^r)$.

Corollary 3.4 *Let $r \geq 3$ and S be some sphere S_n^r with $\Delta = \Delta(S_n^r)$. Then for $p = \omega\left(\Delta^{2r-4} n^{-(r-2)/2}\right)$ the random r-uniform hypergraph $\mathcal{H}^{(r)}(n, p)$ contains a copy of S a.a.s.*

Proof. From Euler's formula it follows that $e_S(v) \leq \frac{2v-4}{r-2}$ and therefore $\gamma(S) = 2/(r-2)$. Since this is an upper bound for the number of r-edges in this induced hypergraph we immideately get $\gamma = 2/(r-2)$. The statement follows now directly from Theorem 1.1. □

3.5 Powers of Hamilton cycles

Consider an $(r-1)$-overlapping Hamilton cycle $C^{(r,r-1)}$ with n vertices which are ordered cyclically. Given an integer i, we define an i-th power $C^{(r)}(i)$ of $C^{(r,r-1)}$ to consist of all r-tuples e such that the maximum distance in this cyclic ordering between any two vertices in e is at most $r+i-2$. In the graph case, the threshold for the appearance of $C^{(2)}(i)$ follows from Riordan's result [24] for $i \geq 3$ (see [18]) and in the case $i=2$ an approximate threshold due to Kühn and Osthus [18] is known. If we count the edges of $C^{(r)}(i)$ by their leftmost vertex we get $e(C^{(r)}(i)) = n\binom{r+i-2}{r-1}$.

Theorem 3.5 *Let $r \geq 3$ and $i \geq 2$ be integers. Suppose that $p = \omega(n^{-1/\binom{r+i-2}{r-1}})$, then the random hypergraph $\mathcal{H}^{(r)}(n,p)$ contains a.a.s a copy of $C^{(r)}(i)$. This threshold is asymptotically optimal.*

Proof. One can argue similarly to Proposition 8.2 in [18] to show $\gamma(C^{(r)}(i)) \leq \binom{r+i-2}{r-1} + O_{r,i}(1/n)$. The statement follows from Theorem 1.1. □

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