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On non-traceable, non-hypotraceable, arachnoid graphs

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Abstract

Motivated by questions concerning optical networks, in 2003 Gargano, Hammar, Hell, Stacho, and Vaccaro defined the notions of spanning spiders and arachnoid graphs. A spider is a tree with at most one branch (vertex of degree at least 3). The spider is centred at the branch vertex (if there is any, otherwise it is centred at any of the vertices). A graph is arachnoid if it has a spanning spider centred at any of its vertices. Traceable graphs are obviously arachnoid, and Gargano et al. observed that hypotraceable graphs (non-traceable graphs with the property that all vertex-deleted subgraphs are traceable) are also easily seen to be arachnoid. However, they did not find any other arachnoid graphs, and asked the question whether they exist. The main goal of this paper is to answer this question in the affirmative, moreover, we show that for any prescribed graph H, there exists a non-traceable, non-hypotraceable, arachnoid graph that contains H as an induced subgraph.

Keywords: spanning spider, arachnoid graph, spanning tree, hamiltonian path, hypotraceable graph

1 Introduction

All graphs considered in this paper are finite, simple, and connected. For a graph G, $V(G)$ and $E(G)$ denotes the set of vertices and the set of edges of G, respectively. Let $X, Y \subseteq V(G)$, $v \in V(G)$. Then $d_G(v)$ is the degree of v in G, $d_G(X,Y)$ denotes the number of edges between X and Y in G, $d_G(X) := d_G(X, V(G) \setminus X)$. The subgraph of G induced by the vertex set X is denoted by $G[X]$ and $G - X := G[V(G) \setminus X]$, $G - v := G - \{v\}$ for any $v \in V(G)$ and for $e \in E(G)$, $G - e$ denotes the graph obtained by deleting e from $E(G)$. $G \cup H$ denotes the disjoint union of graphs G and H. Actually, we also use this notation for the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ if G and H are subgraphs of the same graph.

The leaf number of a graph G, denoted by $l(G)$ is the number of vertices of degree 1 in G. The *minimum leaf number* of a graph G, denoted by $\mathrm{ml}(G)$ is the minimum number of leaves of the spanning trees of G. The *path-covering number* of G, denoted by $\mu(G)$ is the minimum number of vertex-disjoint paths that cover the vertices of G (a path may consist of just one vertex). The *branch number* of G, denoted by $s(G)$ is the minimum number of branch vertices (vertices of degree at least 3) of the spanning trees of G. Each of these graph parameters play an important role in designing cost-efficient optical networks ([6], [2]) and they are all NP-hard to compute, because of their straightforward connection to traceability of graphs. Gargano, Hammar, Hell, Stacho, and Vaccaro [2] defined the notion of *spanning spiders*: these are spanning trees with at most one branch. The spider is *centred* at the branch vertex (if there is any, otherwise it is centred at any of the vertices). They studied the parameter $s(G)$ and graphs with $s(G) \leq 1$. They also defined *arachnoid* graphs; these are graphs that have a spanning spider centred at any of their vertices. Traceable graphs are obviously arachnoid, and Gargano et al. observed that hypotraceable graphs (non-traceable graphs with the property that all vertex-deleted subgraphs are traceable, see [7], [8]) are also easily seen to be arachnoid [2]. However, they did not find any other arachnoid graphs, and asked the question whether they exist. The main goal of this paper is to answer this question in the affirmative, moreover, we show that for any prescribed graph H, there exists a non-traceable, non-hypotraceable, arachnoid graph that contains H as an induced subgraph.

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2 Path-critical graphs

First we construct graphs G for any $\mu \geq 1$ with the property $\mu(G - v) = \mu(G) - 1 = \mu$ for each $v \in V(G)$ (these will be called path-critical graphs). The existence of such graphs is far from from obvious: for $\mu = 1$ these are the hypotraceable graphs, whose existence was an open problem till 1975, when Horton found such a graph on 40 vertices (see [10], [8]) disproving the conjecture of Kapoor, Kronk, and Lick [5]. Actually, even the existence of graphs without concurrent longest paths was an open question from 1966 to 1969 (raised by Gallai [1] and settled by Walther [9]).

For the construction we need the notion of J-cells [4].

Definition 2.1 A pair of vertices (a, b) of a graph G is said to be good if there exists a Hamiltonian path of G between them. A pair of pairs of vertices of G $((a, b), (c, d))$ is said to be good if there exists a spanning subgraph of G consisting of two vertex-disjoint paths, one between a and b and another one between c and d.

Definition 2.2 (Hsu, Lin [4]) The quintuple (H, a, b, c, d) is a J-cell if H is a graph and $a, b, c, d \in V(H)$, such that

(i) The pairs (a, d), (b, c) are good in H.

(ii) None of the pairs (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$, $((a, c), (b, d))$ are good in H.

(iii) For each $v \in V(H)$ there is a good pair in $H - v$ among (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$, $((a, c), (b, d))$.

J-cells can be obtained by deleting two adjacent cubic vertices of a hypohamiltonian graph (non-hamiltonian graph, such that all vertex-deleted subgraphs are hamiltonian, see [3]), as was observed by Thomassen, who used J-cells to construct 3-connected hypotraceable graphs [8]. Here we generalize this construction. The smallest J-cell is obtained from the Petersen graph by deleting two adjacent vertices.

Let $F_i = (H_i, a_i, b_i, c_i, d_i)$ be J-cells for $i = 1, 2, \ldots, k$. Now we define the graphs G_k as follows. G_k consists of vertex-disjoint copies of the graphs $H_1, H_2, \ldots, H_k$, the edges $(b_i, a_{i+1}), (c_i, d_{i+1})$ for all $i = 1, 2, \ldots k - 1$, and the edges $(b_k, a_1), (c_k, d_1)$. We will consider the graphs H_i as (induced) subgraphs of G_k.

Now we explore some useful properties of spanning trees and paths of G_k.

Claim 2.3 *Let T be a spanning tree of G_k. Then there are at most two indices* *i, such that all vertices in $V(H_i)$ has degree 2 in T.*

Proof. Suppose that all vertices in (say) $V(H_1)$ has degree 2 in T. Then $d_T(H_1)$ must be even (since $d_T(H_1) = \sum_{v \in V(H_1)} d(v) - 2|E(T[V(H_1)])| = 2|V(H_1)| - 2|E(T[V(H_1)])|$), thus $d_T(H_1)$ is 2 or 4. If $d_T(H_1) = 2$, then $T[V(H_1)]$ is a hamiltonian path of H_1 and by the second property of J-cells the endvertices of the path are either a_1 and d_1 or b_1 and c_1 (w.l.o.g. assume they are a_1 and d_1). Therefore the edges leaving $V(H_1)$ in T are (b_k, a_1) and (c_k, d_1), thus there are no edges between $V(H_1)$ and $V(H_2)$ in T. If $d_T(H_1) = 4$, then $T[V(H_1)]$ is a spanning subgraph of H_1 consisting of two vertex-disjoint paths. By the second property of J-cells, the endvertices of one of the paths are a_1 and d_1 and the endvertices of the other path are b_1 and c_1. Thus in this case there is no path between a_1 and b_1 in $T[V(H_1)]$. It is clear now that if there is an index $i \neq 1, 2$, such that all vertices in $V(H_i)$ has degree 2 in T, then T is not connected, a contradiction. □

Claim 2.4 *Let $l \geq 2$. Then $ml(G_{2l+1}) \geq l + 1$.*

Proof. Assume to the contrary that G_{2l+1} has a spanning tree T with at most l leaves. Then the number of vertices of degree at least 3 in T is at most $l-2$, thus the number of vertices not having degree 2 is at most $2l-2$. This means that there are at least three indices i, such that $V(H_i)$ only contains vertices of degree 2 in T, a contradicition by Claim 2.3. □

Claim 2.5 *G_4 has a hamiltonian path P, such that there is no edge of P between H_1 and H_4 and for any vertex $v \in V(G_5)$ there is a hamiltonian path P of $G_5 - v$, such that there is no edge of P between H_1 and H_5.*

Proof. The first part of the claim is easy to see: there is a hamiltonian path of H_i between b_i and c_i and a hamiltonian path of H_{i+1} between a_{i+1} and d_{i+1}, by the first property of J-cells, thus $H_1 \cup H_2$ and $H_3 \cup H_4$ are hamiltonian, therefore there is a hamiltonian path P_1 of $H_1 \cup H_2$ starting at b_2 and a hamiltonian path P_3 of $H_3 \cup H_4$ starting at a_3. Now $E(P_1) \cup (b_2, a_3) \cup E(P_3)$ is a hamiltonian path of G_4 without edges between H_1 and H_4. Let now $F = (H, a, b, c, d)$ be any of the J-cells used in the construction of G_5 and let us check whether (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$, or $((a, c), (b, d))$ is good in $H - v$. Let us number the J-cells used to construct G_5, such that $H_3 = H$ in the first four cases, and $H_2 = H$ in the last two cases. If $(a, b) = (a_3, b_3)$ is good in $H_3 - v$, then let P be a hamiltonian path of $H_3 - v$ between a_3 and b_3. We have seen that $H_i \cup H_{i+1}$ is hamiltonian, therefore $H_i \cup H_{i+1}$ has a hamiltonian path starting at any of its vertices. Let P_1 be a hamiltonian path of $H_1 \cup H_2$ starting at b_2 and let P_4 be a hamiltonian path of $H_4 \cup H_5$ starting at a_4. Then $E(P_1) \cup (b_2, a_3) \cup E(P) \cup (b_3, a_4) \cup E(P_4)$ is the edge set of a hamiltonian path of $G_5 - v$ and does not contain any edges between H_1 and H_5. The cases when (a, c), (b, d), or (c, d) is good is dealt with similarly. If $((a, b), (c, d)) = ((a_2, b_2), (c_2, d_2))$ is good in $H_2 - v$, then let Q be the union of the vertex-disjoint $a - b$ and $c - d$ paths that cover all vertices of $H_2 - v$. Let furthermore Q_1 be a hamiltonian path between b_1 and c_1 in H_1, and Q_3 be a hamiltonian path between d_3 and either b_3 or c_3 (say w.l.o.g. b_3) in $G_3 - a_3$. Q_1 and Q_3 exist since F_1 and F_3 are J-cells. Then $E(Q_1) \cup (b_1, a_2) \cup (c_1, d_2) \cup E(Q) \cup (b_2, a_3) \cup (c_2, d_3) \cup E(Q_3) \cup (b_3, a_4) \cup E(P_4)$ is again the edge set of a hamiltonian path of $G_5 - v$ that does not contain any edges between H_1 and H_5. The case when $((a, c), (b, d))$ is good is dealt with similarly. □

Theorem 2.6 *For any $v \in V(G_{4k+5})$ we have $\mu(G_{4k+5} - v) = \mu(G_{4k+5}) - 1 = k + 1$, thus G_{4k+5} is path-critical for $k \geq 1$.*

Proof. Let us denote $G_{4k+5}[\cup_{i=n}^{m} V(H_i)]$ by $G(n, m)$ for $1 \leq n < m \leq 4k + 5$. It is obvious that if $n \neq 1$ or $m \neq 4k + 5$, then $G(n, m)$ is isomorphic to some graph $G_{m-n+1} - (b_{m-n+1}, a_1) - (c_{m-n+1}, d_1)$, thus $G(n, m)$ is traceable if $m = n + 3$ and $G(n, m) - v$ is traceable for any $v \in G(n, m)$ if $m = n + 4$ by Claim 2.5. Since $G(1, 4), G(5, 8), \ldots, G(4k - 3, 4k)$ and $G(4k + 1, 4k + 5) - v$ are all traceable, the vertices of $G_{4k+5} - v$ can be covered by $k + 1$ vertex-disjoint paths, that is $\mu(G_{4k+5} - v) \leq k + 1$ for any $v \in V(G)$. On the other hand, we show that $\mu(G_{4k+5}) \geq k + 2$. Assume to the contrary that there are at most $k + 1$ vertex-disjoint paths that cover the vertices of G_{4k+5}. Since G_{4k+5} is connected, it is possible to add some (at most k, but it is irrelevant) edges to these paths to obtain a spanning tree of G_{4k+5} with at most $2k + 2$ leaves. On the other hand, by Lemma 2.4, $\mathrm{ml}(G_{4k+5}) \geq 2k + 3$, a contradiction. Since for any graph G, $\mu(G) \leq \mu(G - v) + 1$ is obvious, we have $k + 1 \leq \mu(G_{4k+5}) - 1 \leq \mu(G_{4k+5} - v) \leq k + 1$, and the theorem is proved. □

The graphs G_k possess some other interesting properties; these are omitted here, due to lack of space.

3 Arachnoid graphs

Now it is not difficult to find non-traceable, non-hypotraceable, arachnoid graphs. Let G_k^j be the graph obtained from G_k by adding j new vertices $u_1, u_2, \ldots, u_j$ and edges between u_i and every vertex of G_k to G_k for $i = 1, 2, \ldots, j$.

Theorem 3.1 *G^k_{4k+5} is an arachnoid graph that is neither traceable, nor hypotraceable for any $k \geq 1$.*

Proof. Let $G = G^k_{4k+5}$. We have to show that for any $w \in V(G)$, G has a spanning spider centred at w. Let v be a neighbour of w, such that $v \in G_{4k+5}$ (such a v clearly exists). Now by Theorem 2.6, the vertices of $G_{4k+5} - v$ can be covered by $k + 1$ vertex-disjoint paths, thus using the vertices $u_1, \ldots, u_k$ (that are all connected to all vertices of G_{4k+5}) a hamiltonian path of $G - v$ is easy to obtain. Now by adding the edge (v, w) to this path we obtain a spanning spider of G centred at w, therefore G is arachnoid, indeed.

Now we show that G is not traceable. Assume to the contrary that there exists a hamiltonian path P of G and let us delete the vertices $u_1, \ldots, u_k$ from P. We obtain at most $k + 1$ vertex-disjoint paths, such that they cover the vertices of G_{4k+5}, which is a contradiction, by Theorem 2.6.

Finally, we have to show that G is not hypotraceable. It is easy to see that $G - u_i$ is not traceable, the proof is the same as the proof of the non-traceablity of G (by deleting the u_i's we would obtain at most k paths, instead of at most $k + 1$). □

It is easy to see that adding any edges between the u_i's does not make the graph either traceable or hypotraceable (while the arachnoid property is obviously preserved), therefore we can obtain a non-traceable, non-hypotraceable, arachnoid graph that contains any prescribed graph H as an induced subgraph.

Gargano et al. also proposed the more general problem whether there exist arachnoid graphs containing a vertex v, such that v is the center of only spanning spiders S, for which $d_S(v) \geq 4$. This question is still open. Now that we have seen new arachnoid graphs, it is worth asking whether there are arachnoid graphs containing *several* vertices v, such that v is the center of only spanning spiders S, for which $d_S(v) \geq d$ for some fixed $d \geq 4$.

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