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Path decompositions of regular graphs with prescribed girth ⋆

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Abstract

A P_ℓ-decomposition of a graph G is a set of pairwise edge-disjoint paths of G with ℓ edges that cover the edge set of G. Kotzig (1957) proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching, and also asked what are the necessary and sufficient conditions for an ℓ-regular graph to admit a P_ℓ-decomposition, for odd ℓ. Let g, ℓ and m be positive integers with $g \geq 3$. We prove that, (i) if ℓ is odd and $m > 2\lfloor(\ell-2)/(g-2)\rfloor$, then every $m\ell$-regular graph with girth at least g that contains an m-factor admits a P_ℓ-decomposition; (ii) if $m > \lfloor(\ell-2)/(g-2)\rfloor$, then every $2m\ell$-regular graph with girth at least g admits a P_ℓ-decomposition.

Keywords: regular graphs, edge-decomposition, girth, path decomposition

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1 Introduction

A set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of pairwise edge-disjoint subgraphs of a graph G is called a *decomposition* of G if these subgraphs cover the edge set of G. If H_i, for $1 \leq i \leq k$, is isomorphic to a graph H, then we say that $\mathcal{D}$ is an H*-decomposition* of G. When H is a path of length 2, it is easy to prove that a connected graph G admits an H-decomposition if and only if G has an even number of edges. On the other hand, Dor and Tarsi [9] proved that deciding whether a graph admits an H-decomposition is an NP-complete problem whenever H has a component with at least 3 edges. It is then natural to look for sufficient conditions for a graph G to admit an H-decomposition. When H is a tree, Barát and Thomassen [1] conjectured that high edge-connectivity (together with the obvious divisibility condition on the number of edges) may suffice. In this paper we focus on the special case where G is an ℓ-regular graph, and H is a path of length ℓ (ℓ edges), which we denote by P_ℓ.

Kotzig [16] and also Bouchet and Fouquet [6] independently proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching. Kotzig asked what are the necessary and sufficient conditions for an odd ℓ-regular graph G to be decomposable into paths of length ℓ. A necessary condition is that G admits a decomposition into an $\big((\ell-1)/2\big)$-factor and an $\big((\ell+1)/2\big)$-factor. Favaron, Genest, and Kouider [10] proved that this condition is not sufficient, and conjectured that the existence of a perfect matching may suffice for $\ell \geq 5$. Favaron, Genest, and Kouider [10] proved that it is sufficient for a 5-regular to contain a perfect matching and no cycles of length four to admit a P_5-decomposition. More recently, it was proved [5] that every triangle-free 5-regular graph containing a perfect matching admits a P_5-decomposition. Many surveys and books on graph decompositions have appeared in the literature [2,7,11,12]. In what follows we will restrict our attention to decompositions of regular graphs. In [14] it is proved that every 4-regular bipartite graph admits a P_4-decomposition. For other results concerning $2k$-regular graphs and cartesian product of regular graphs, the reader is referred to [15,18]; and for results on decompositions of regular graphs with large girth, we mention Kouider and Lonc [17].

Let $\mathcal{D}$ be a decomposition of a graph G into paths. Given a vertex v of G, we denote by $\mathcal{D}(v)$ the number of elements of $\mathcal{D}$ containing v as an end-vertex. We say that $\mathcal{D}$ is *balanced* if $\mathcal{D}(u) = \mathcal{D}(v)$ for every $u, v \in V(G)$. Heinrich, Liu and Yu [13] proved that if G is a $3m$-regular graph that contains an m-factor, then G admits a balanced P_3-decomposition. Our main contributions (Theorems 3.2 and 3.4) are as follows. Let g, ℓ and m be positive integers with $g \geq 3$. (i) If ℓ is odd and $m > 2\lfloor(\ell-2)/(g-2)\rfloor$, then every $m\ell$-regular graph with girth at least g that contains an m-factor admits a balanced P_ℓ-decomposition. (ii) If $m > \lfloor(\ell-2)/(g-2)\rfloor$, then every $2m\ell$-regular graph with girth at least g admits a balanced P_ℓ-decomposition.

Our proofs require a generalization of the technique used by the authors in [4]. Owing to space limitation, we present only sketches of some of the proofs. The basic terminology and notation used in this paper are standard (see, e.g. [8]).

2 The Disentangling Lemma

We present three lemmas that together form the central part of the proof of our main result. Among them, Lemma 2.3, which we call Disentangling Lemma, is the most important result in the proof of our main results. We first introduce some definitions and establish the notation. A *path* P in G is a sequence of distinct vertices $P = v_0v_1\cdots v_\ell$ such that $v_iv_{i+1} \in E(G)$, for $i = 0, 1, \ldots, \ell-1$. The vertices v_0 and v_ℓ are the *end-vertices* of P. The *length* of P is the number of its edges. Sometimes, it is convenient to consider that a path $P = v_0v_1\cdots v_\ell$ is the subgraph of G induced by the edges v_iv_{i+1} for $i = 0, \ldots, \ell-1$. A *vanilla trail* is a trail $T = v_0v_1\cdots v_\ell$ such that $v_1\cdots v_{\ell-1}$ is a path. A *vanilla ℓ-trail* is a vanilla trail of length ℓ.

Let $\mathcal{D}$ be a decomposition of a graph G into trails. Let v be a vertex of G and let $uv \in E(G)$ and $T \in \mathcal{D}$ be such that $uv \in E(T)$. If $T = x_0x_1\cdots x_\ell$ with $x_0 = u$ and $x_1 = v$, we say that uv is a *pre-hanging edge* of $\mathcal{D}$ at v. We denote by $\mathrm{preHang}(v, \mathcal{D})$ the number of pre-hanging edges of $\mathcal{D}$ at v. We say that $\mathcal{D}$ is *k-pre-complete* if $\mathrm{preHang}(v, \mathcal{D}) > k$, for all v in $V(G)$. If $d_T(u) = 1$, then we say that uv is a *hanging edge* of $\mathcal{D}$ at v. We denote by $\mathrm{Hang}(v, \mathcal{D})$ the number of hanging edges of $\mathcal{D}$ at v. We say that $\mathcal{D}$ is *k-complete* if $\mathrm{Hang}(v, \mathcal{D}) > k$, for all v in $V(G)$.

Let $\mathcal{D}$ be a decomposition of a graph G into trails. We extend to trails the definition of $\mathcal{D}(v)$ given in Section 1 for paths. Given a vertex v of G, we denote by $\mathcal{D}(v)$ the number of elements of $\mathcal{D}$ that have v as an end-vertex. If an element T of $\mathcal{D}$ is such that v is the end-vertex of T, then we count T twice (or with multiplicity 2) in $\mathcal{D}(v)$. Analogously, we say that $\mathcal{D}$ is *balanced* if $\mathcal{D}(u) = \mathcal{D}(v)$ for every $u, v \in V(G)$. Lemma 2.1 shows how to obtain a balanced complete vanilla ℓ-trail decomposition from some balanced pre-complete vanilla ℓ-trail decomposition.

Lemma 2.1 *Let g, k, ℓ, r be positive integers with $g \geq 3$ and let G be a graph* *with girth at least g. If $r \geq \lfloor(\ell-2)/(g-2)\rfloor$ and G admits a balanced $(k+r)$-pre-complete vanilla ℓ-trail decomposition $\mathcal{D}$, then G admits a balanced k-complete vanilla ℓ-trail decomposition $\mathcal{D}'$.*

Proof Let $\mathcal{D}$ be a balanced $(k+r)$-pre-complete vanilla ℓ-trail decomposition of G that maximizes $\sum_{v\in V(G)} \mathrm{Hang}(v,\mathcal{D})$. We claim that $\mathcal{D}$ is k-complete, i.e, $\mathrm{Hang}(v,\mathcal{D}) > k$ for each vertex v of G.

Suppose, by contradiction, that $\mathcal{D}$ is not k-complete. Then there is a vertex v of G such that $\mathrm{Hang}(v,\mathcal{D}) \leq k$. Since $\mathcal{D}$ is $(k+r)$-pre-complete, $\mathrm{preHang}(v,\mathcal{D}) \geq k+r+1$. Thus, there are at least $r+1$ pre-hanging edges at v that are not hanging edges at v, say $x_1v,\ldots,x_{r+1}v$. Let $T_1 = y_0y_1\cdots y_\ell$ be the element of $\mathcal{D}$ that contains x_1v, where $y_0 = x_1$ and $y_1 = v$ and let $X = \{x_1,\ldots,x_{r+1}\}$. Let $x'_1,\ldots,x'_s$ be the vertices of X contained in $V(T_1)$, ordered by distance from y_1 in $T_1 - y_0y_1$. Let l_0 be the distance from y_1 to x'_1 in T_1, and let l_i be the distance from x'_i to x'_{i+1} in T_1, for $0 < i \leq s-1$. Since the girth of G is at least g, we have $l_0 \geq g-1$, and $l_i \geq g-2$ for $1 \leq i \leq s-1$. Since $x'_1,\ldots,x'_s$ are ordered by distance from y_1 in $T_1 - y_0y_1$, we have $\ell - 1 \geq \sum_{i=0}^{s-1} l_i \geq g-1+(s-1)(g-2) = s(g-2)+1$. Therefore, $s \leq (\ell-2)/(g-2)$, which implies $s \leq r$. Since $|X| = r+1 > s$, there is at least one vertex in X, say x_p, that is not a vertex of T_1.

Let T_p be the element of $\mathcal{D}$ that contains x_pv. Let $T'_1 = T_1 - x_1v + x_pv$ and $T'_p = T_p - x_pv + x_1v$, and let $\mathcal{D}' = \mathcal{D} - T_1 - T_p + T'_1 + T'_p$. It is not hard to check that D' is balanced. Since $x_p \notin V(T_1)$, we have $d_{T'_1}(x_p) = 1$, which implies that x_pv is a hanging edge of $\mathcal{D}'$ at v. Therefore, $\sum_{v\in V(G)} \mathrm{Hang}(v,\mathcal{D}') > \sum_{v\in V(G)} \mathrm{Hang}(v,\mathcal{D})$, a contradiction to the maximality of $\sum_{v\in V(G)} \mathrm{Hang}(v,\mathcal{D})$. □

The next result is an essential tool in the proof of Lemma 2.3.

Lemma 2.2 *Let ℓ and g be positive integers such that $g \geq 3$ and let G be a graph with girth at least g. Let $\mathcal{D}$ be a decomposition of G. Let T be a trail of length ℓ in G, and let v be an internal vertex of T. If* $\mathrm{Hang}(v,\mathcal{D}) > \lfloor(\ell-2)/(g-2)\rfloor$*, then there is a hanging edge uv of $\mathcal{D}$ at v such that $u \notin V(T)$.*

Proof We start by splitting T at v, obtaining two trails T_1, T_2 of length ℓ_1 and ℓ_2, respectively. Let $x_1v,\ldots,x_{k+1}v$ be distinct hanging edges at v. Put $X = \{x_1,\ldots,x_{k+1}\}$. Denote by s_1 the number of vertices of X in T_1. Let $x'_1,\ldots,x'_{s_1}$ be the vertices of X contained in $V(T_1)$, ordered by distance from v in T_1.

Suppose $s_1 \geq 1$. Let l_0 be the distance in T_1 from v to x'_1, and l_i be the distance in T_1 from x'_i to x'_{i+1}, for $i > 0$. Since G has girth at least g, we have $l_0 \geq g-1$, and $l_i \geq g-2$ for $1 \leq i \leq s_1 - 1$. Therefore, $\ell_1 \geq \sum_{i=0}^{s-1} l_i \geq g-1+(s_1-1)(g-2) = s_1(g-2)+1$. Defining s_2 (with respect to T_2) analogously to s_1, we obtain that if $s_2 \geq 1$, then $\ell_2 \geq s_2(g-2)+1$. Therefore, if $s_1, s_2 \geq 1$, we have $s_1 + s_2 \leq (\ell-2)/(g-2)$. Now suppose $s_1 = 0$. Since v is an internal vertex of T, we have $\ell - 1 \geq \ell_2 \geq s_2(g-2)+1$, which implies that $s_1 + s_2 = s_2 \leq (\ell-2)/(g-2)$. In both cases, we have $s_1 + s_2 \leq \lfloor (\ell-2)/(g-2) \rfloor = k$. Since $|X| = k+1 > s_1 + s_2$, there is at least one element u in X that is not contained in T. □

Now we are ready to state and give a sketch of the proof of the Disentangling Lemma.

Lemma 2.3 (Disentangling Lemma) *Let g and ℓ be positive integers with $g \geq 3$ and let G be a graph of girth g. Let $k \geq \lfloor (\ell-2)/(g-2) \rfloor$. If G admits a balanced k-complete decomposition $\mathcal{D}$ into vanilla ℓ-trails, then G admits a balanced k-complete P_ℓ-decomposition .*

Sketch of the proof. For every vanilla trail T in G, let $\tau(T)$ be the number of end-vertices of T that have degree greater than 1 in T. Let $\mathcal{D}^*$ be a balanced k-complete decomposition into vanilla ℓ-trails that minimizes $\sum_{T\in\mathcal{D}} \tau(T)$.

Suppose that there is a vanilla trail T_0 in $\mathcal{D}^*$ that is not a path. Let x be an end-vertex of T_0 of degree greater than 1 in T_0, and let C be a cycle in T_0 that contains x. Let y be a neighbour of x in C. By Lemma 2.2, there is a hanging edge uy of $\mathcal{D}^*$ at y such that $u \notin V(T_0)$. Let T_1 be the element of $\mathcal{D}^*$ that contains uy. Now, let $T_0' = T_0 - xy + uy$, $T_1' = T_1 - uy + xy$, and put $\mathcal{D}' = \mathcal{D}^* - T_0 - T_1 + T_0' + T_1'$. Note that $\mathcal{D}'(v) = \mathcal{D}^*(v)$ for every v in $V(G)$, and $\tau(T_0') = \tau(T_0) - 1$. If $\tau(T_1') \leq \tau(T_1)$, then $\mathcal{D}'$ is a balanced k-complete decomposition of G into vanilla ℓ-trails such that $\sum_{T\in\mathcal{D}'} \tau(T) < \sum_{T\in\mathcal{D}^*} \tau(T)$. Otherwise, we would have $\tau(T_1') = \tau(T_1)+1$ and T_1' contains a cycle C' that contains xy. Let y' be a neighbour of x in C' such that $y' \neq y$. Now, repeat the above operation, as long as necessary, considering T_1' and y' instead of T_0 and y. We can show that such a procedure halts, and we obtain the desired decomposition, concluding the proof.

3 Main results

In this section we show how to use Petersen's Factorization Theorem to obtain some complete decomposition of a graph G into vanilla ℓ-trails. Then, we use Lemma 2.3 to obtain a decomposition into paths of length ℓ. Theorem 3.1 guarantees that the statement of the Theorem 3.2 holds for $\ell = 3$, while Proposition 3.3 guarantees that the statement of Theorem 3.4 holds for $\ell = 2$.

Theorem 3.1 (Heinrich–Liu–Yu [13]) *If G is a $3m$-regular graph that contains an m-factor, then G admits a balanced P_3-decomposition.*

Theorem 3.2 *Let ℓ, g and m be positive integers such that ℓ is odd and $g \geq 3$, and let G be an $m\ell$-regular graph with girth at least g that contains an m-factor. If $m > 2\lfloor(\ell-2)/(g-2)\rfloor$, then G admits a balanced P_ℓ-decomposition.*

Proof Suppose the statement is false, and let ℓ be the smallest integer such that the statement is not true. By Theorem 3.1, the statement is true for $\ell = 3$ and $g \geq 3$. Thus, we may assume that $\ell \geq 5$.

Let M be an m-factor of G. By Petersen's Theorem, $G - E(M)$ admits a 2-factorization $\{F_1, \ldots, F_{(\ell-1)m/2}\}$. Let H be the union of m of these factors, i.e, H is a $2m$-factor of G. Then $G' = G - E(H)$ is a $(\ell-2)m$-regular graph with girth at least g. Note that, $m \geq 2\lfloor(\ell-2)/(g-2)\rfloor > 2\lfloor(\ell-4)/(g-2)\rfloor$. Thus, by the minimality of ℓ, G' admits a balanced decomposition $\mathcal{D}'$ into paths of length $\ell - 2$.

We claim that $\mathcal{D}'(v) = m$ for every vertex v in $V(G)$. Let $|V(G)| = n$. Note that G' contains $n(\ell-2)m/2$ edges, hence $\mathcal{D}'$ contains $nm/2$ paths. Thus $\sum_{v\in V(G)} \mathcal{D}'(v) = nm$. Since $\mathcal{D}'$ is balanced, we have $\sum_{v\in V(G)} \mathcal{D}'(v) = n\mathcal{D}'(u)$, for every u in $V(G)$. Therefore, $\mathcal{D}'(v) = m$ for every vertex v of G. Choose an Eulerian orientation for H. Note that $d^+_H(v) = m = \mathcal{D}'(v)$. Thus, we can extend each path P' of $\mathcal{D}'$ to a vanilla ℓ-trail T by adding to P' one edge of H at each of its end-vertices. Let $\mathcal{D}$ be the decomposition obtained. Note that $\mathcal{D}(v) = d^-_H(v) = m$ for every vertex v in $V(G)$, because H has an Eulerian orientation. Therefore, $\mathcal{D}$ is balanced. Put $r = \lfloor(\ell-2)/(g-2)\rfloor$ and $k = m - r - 1$. Note that $k \geq \lfloor(\ell-2)/(g-2)\rfloor$. By the definition of pre-hanging edge, the edges of H that leave a fixed vertex v of G are precisely the pre-hanging edges of $\mathcal{D}$ at v; hence, $\mathcal{D}$ is $(m-1)$-pre-complete. By Lemma 2.1 applied with g, k, ℓ, and r, the graph G admits a balanced k-complete decomposition $\mathcal{D}''$ into vanilla ℓ-trails. Since $k \geq \lfloor(\ell-2)/(g-2)\rfloor$, by Lemma 2.3, G admits a balanced k-complete P_ℓ-decomposition. □

The proof of Theorem 3.4 is similar to the proof of Theorem 3.2, and uses Proposition 3.3 instead of Theorem 3.1.

Proposition 3.3 *If G is a $4m$-regular graph, then G admits a balanced P_2-decomposition.*

Proof Consider an Eulerian orientation of G. Since G is $4m$-regular, we have $d^+(v) = d^-(v) = 2m$ for every $v \in V(G)$. For each vertex v of G, decompose the set of edges corresponding to the directed edges that leave v

into m paths of length 2. Let $\mathcal{D}$ be the obtained decomposition, and note that $\mathcal{D}(v) = d^-(v) = 2m$ for every vertex v of $V(G)$. $\square$

Theorem 3.4 *Let ℓ, g and m be positive integers such that $g \geq 3$ and let G be a $2m\ell$-regular graph with girth at least g. If $m > \lfloor (\ell - 2)/(g - 2) \rfloor$, then G admits a balanced P_ℓ-decomposition.*

4 Concluding remarks

We believe that the Disentangling Lemma (Lemma 2.3) can be used to obtain other decomposition results. In fact, in a forthcoming paper [3], we use a version of the Disentangling Lemma for bipartite graphs to prove that, for every positive integer ℓ, there exists a constant $k = k(\ell)$, such that if G is k-edge connected and ℓ divides $|E(G)|$, then G admits a P_ℓ-decomposition. This result solves for path decomposition the Conjecture of Barát and Thomassen [1] on tree decomposition.

For the special case of Theorem 3.2, in which $g = \ell - 1$, we use a generalization of the techniques in [5] to prove that the following holds for odd ℓ: Every ℓ-regular graph with girth $\ell - 1$ that contains a perfect matching admits a P_ℓ-decomposition.

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