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An universality argument for graph homomorphisms [1]

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1 Introduction

It is a non-trivial result that every countable partial order can be found as a suborder of the homomorphism order of graphs. This has been first proved in the even stronger setting of category theory [12]. Subsequently, it has been shown that many restricted classes of graphs (such as oriented trees

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[9], oriented paths [8], partial orders and lattices [10]) admit this universality property.

We show a very simple and versatile argument based on divisibility which immediately yields the universality of the homomorphism order of directed graphs and discuss three applications.

2 Universal partial orders

In this section we give a construction of a universal partial order. Let us first review some basic concept and notations.

In the whole paper we consider only finite and countable partial orders. An *embedding* of a partial order $(Q, \leq_Q)$ in $(P, \leq_P)$ is a mapping $e : P \to Q$ satisfying $x \leq_P y$ if and only if $e(x) \leq_Q e(y)$. In such a case we also say that $(Q, \leq_Q)$ is a *suborder* of $(P, \leq_P)$.

For a given partial order $(P, \leq)$, the *down-set* $\downarrow x$ is $\{y \in P \mid y \leq x\}$. Similarly, the *up-set* is $\uparrow x = \{y \in P \mid x \leq y\}$.

Any finite partial order $(P, \leq_P)$ can be represented by finite sets ordered by the inclusion, e.g. when x is represented by $\downarrow x$. This is a valid embedding, because $\downarrow x \subseteq \downarrow y$ if and only if $x \leq_P y$.

Without loss of generality we may assume that P a subset of some fixed countable set A, e.g. $\mathbb{N}$. Consequently, the partial order formed by the system $P_{\mathrm{fin}}(A)$ of all finite subsets of A ordered by the inclusion contains any finite partial order as a suborder. Such orders are called are *finite-universal*. We reserve the term *universal* for orders that contain every countable partial order as a suborder.

Finite-universal and universal orders may be viewed as countable orders of rich structure — they are of infinite dimension, and that they contain finite chains, antichains and decreasing chains of arbitrary length. While finite-universal partial orders are rather easy to construct, e.g., as the disjoint union of all finite partial orders, the existence of a universal partial order can be seen as a counter-intuitive fact: there are uncountably many countable partial orders, yet all of them can be "packed" into a single countable structure.

The universal partial order can be build in two steps. For these we need further terminology: An order is *past-finite*, if every down-set is finite. An order is *past-finite-universal* if it contains every past-finite order. Analogously, *future-finite* and *future-finite-universal* orders are defined w.r.t. finiteness of up-sets.

1. Observe that the mapping $e(x) = \downarrow x$ is also an embedding $e : (P, \leq) \to$

$(P_{\text{fin}}(A), \subseteq)$ in the case when $(P, \leq)$ is past-finite and $P \subseteq A$. Since a past-finite partial order turns to be future-finite when the direction of inequalities is reversed, we get:

Proposition 2.1 *For any countably infinite set A it holds that*

(i) the order $(P_{\text{fin}}(A), \subseteq)$ is past-finite-universal, and

(ii) the order $(P_{\text{fin}}(A), \supseteq)$ is future-finite-universal.

2. For a given partial order $(Q, \leq)$ we construct the *subset order*, $(P_{\text{fin}}(Q), \leq^{\text{dom}}_Q)$, where

$$X \leq^{\text{dom}}_Q Y \iff \text{for every } x \in X \text{ there exists } y \in Y \text{ such that } x \leq y.$$

We show that the subset order is universal:

Theorem 2.2 *For every future-finite-universal partial order $(F, \leq_F)$ it holds that $(P_{\text{fin}}(F), \leq^{\text{dom}}_F)$ is universal.*

Proof (sketch). It is easy to check that $(P_{\text{fin}}(F), \leq^{\text{dom}}_F)$ is indeed partial order. We sketch the way to embed any given partial order in $(P_{\text{fin}}(F), \leq^{\text{dom}}_F)$. Let be given any countable partial order $(P, \leq_P)$. Without loss of generality we may assume that $P \subseteq \mathbb{N}$. This way we enforce a linear order $\leq$ on the elements of P. The order $\leq$ is unrelated to the partial order $\leq_P$. We decomposed $(P, \leq_P)$ into:

(i) The *forward order* $\leq_f$, where $x \leq_f y$ if and only if $x \leq_P y$ and $x \leq y$, and

(ii) the *backward order* $\leq_b$, where $x \leq_b y$ if and only if $x \leq_P y$ and $x \geq y$.

For every $x \in P$ both sets $\{y \mid y \leq_f x\}$ and $\{y \mid x \leq_b y\}$ are finite. In other words $(P, \leq_f)$ is past-finite and $(P, \leq_b)$ is future-finite.

Since $(F, \leq_F)$ is future-finite-universal, there is an embedding $e : (P, \leq_b) \to (F, \leq_F)$. For every $x \in P$ we now define: $g(x) = \{e(y) \mid y \leq_f x\}$. $\square$

An example of this construction is depicted in Figure 1. We chose F to be set of prime numbers for reasons that will become clear shortly. We remark that the embedding g constructed in the proof of Theorem 2.2 has the property that $g(x)$ depends only on elements $y < x$. Such embeddings are known as on-line embeddings because they can be constructed inductively without a-priori knowledge of the whole partial order. See also [8,9,6] for similar constructions.

By Proposition 2.1 we see that a particular example of a past-finite-universal order is $(P_{\text{fin}}(\mathbb{P}), \subseteq)$, where $\mathbb{P}$ is the class of all odd prime numbers. It follows that $(P_{\text{fin}}(\mathbb{P}), \supseteq)$ is future-finite-universal. As for $X, Y \in P_{\text{fin}}(\mathbb{P})$ holds that

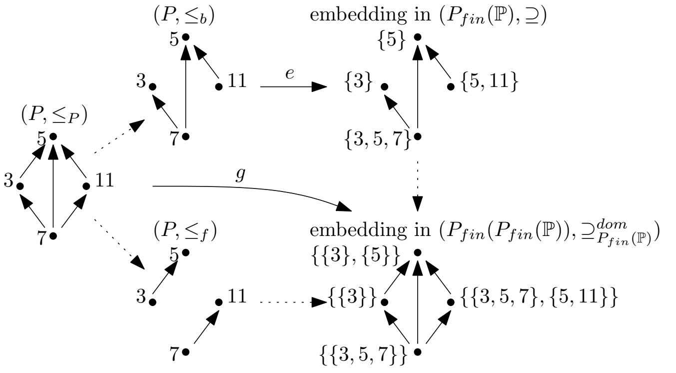

Fig. 1. A representation of $(P, \leq_P)$ according to Theorem 2.2

$X \subseteq Y$ if and only if $\prod X$ divides $\prod Y$, we immediately obtain a special embeddings of the subset orders by divisibility as:

Proposition 2.3

a) The divisibility order $(\mathbb{N}, |)$ *is past-finite-universal,*

b) the reversed divisibility order $(\mathbb{N}, \overleftarrow{|}\,)$ *is future-finite-universal,*

c) the subset reverse divisibility order $(P_{\text{fin}}(\mathbb{N}), \overleftarrow{|}^{dom}_{\mathbb{N}})$ *is universal.*

In the following we show that the subset reverse divisibility order can be directly represented in the homomorphism order.

3 The homomorphism order

For given directed graphs G and H a *homomorphism* $f : G \to H$ is a mapping $f : V_G \to V_H$ such that $(u, v) \in V_G$ implies $(f(u), f(v)) \in V_H$. He existence of homomorphism $f : G \to H$ is traditionally denoted by $G \to H$. This allows us to consider the existence of a homomorphism, $\to$, to be a (binary) relation on the class of directed graphs.

The relation $\to$ is reflexive (the identity is a homomorphism) and transitive (a composition of two homomorphisms is still a homomorphism). Thus the existence of a homomorphism induces a quasi-order on the class of all finite directed graphs. We denote the quasi-order induced by the existence of homomorphisms on directed graph by $(\mathsf{DiGraphs}, \leq)$ and on undirected graphs by $(\mathsf{Graphs}, \leq)$. When speaking of orders, we use $G \leq H$ in the same sense as $G \to H$. These quasi-orders can be easily transformed into a partial order by choosing a particular representative for each equivalence class. In the case of graph homomorphism such representative is up to isomorphism unique vertex minimal element of each class, the *graph core*.

Both homomorphism orders $(\mathsf{DiGraphs}, \leq)$ and $(\mathsf{Graphs}, \leq)$ have been extensively studied and proved to be fruitful areas of research, see [5].

The original argument for universality of partial order [12] used complex graphs and ad-hoc constructions. It thus came as a surprise that the homomorphism order is universal even on the class of oriented paths [6]. While oriented paths is a very simple class of graphs, the universality argument remained rather complex. We can show show the universality of another restricted class easily.

Let $\overrightarrow{C}_k$ stand for the directed cycle on k vertices with edges oriented in the same direction; $\mathsf{DiCycle}$ is the class of directed graphs formed by all $\overrightarrow{C}_k$, $k \geq 3$; and $\mathsf{DiCycles}$ is the class of directed graphs formed by disjoint union of finitely many graphs in $\mathsf{DiCycle}$.

Theorem 3.1 *The partial order* $(\mathsf{DiCycles}, \leq)$ *is universal.*

Proof. As $\overrightarrow{C}_k \to \overrightarrow{C}_l$ if and only if $k \overleftarrow{\mid} l$, we get the conclusion directly from Proposition 2.3. $\square$

4 Applications

4.1 The fractal property of the homomorphism order

As a strengtening of the universality of homomorphism order we can show that every non-trivial interval in the order is in universal. This property under name of fractal property was first shown by Nešetřil [11] but the proof was difficult and never published. Easier proof based on the divisibility argument will appear in [7].

4.2 Universality of order induced by locally injective homomorphisms

Graph homomorphisms are just one of many mappings between graphs that induce a partial order. Monomorphisms, embeddings or full homomorphisms also induce partial orders. The homomorphism order however stands out as especially interesting and the universality result is one of unique properties of it. Other orders fails to be universal for rather trivial reasons, such as lack of infinite increasing or decreasing chains. A notable exception is the graph minor order, that is known to not be universal as a consequence of celebrated result of Robertson and Seymour [14]. We consider the following order:

A homomorphism $f : G \to H$ is *locally injective*, if for every vertex v the restriction of the mapping f to the domain $N_G(v)$ and range $N_H(f(v))$ is injective. (Here $N_G(v)$ denote the open neighborhood of a vertex). This order was first studied by Fiala, Paulusma and Telle in [4] where the degree refinement matrices are used to describe several interesting properties. We can further show:

Theorem 4.1 *The class of all finite connected graphs ordered by the existence of locally injective homomorphisms is universal.*

The proof of this theorem is based on a simple observation that every homomorphism between directed cycles is also locally injective homomorphism. The universality of locally injective homomorphism order on DiCycles thus follows from Theorem 3.1. This is a key difference between Theorem 3.1 and the universality of oriented paths: homomorphisms between oriented paths require flipping that can not be easily interpreted by locally injective homomorphisms.

In the second part of proof of Theorem 4.1 the cycles need to be connected together into a single connected graph in a way preserving all homomorphisms intended. This argument is technical and will appear in [1].

4.3 Universality of homomorphism order of line graphs

We close the paper by yet another application answering question of Roberson [13] asking about the universality of homomorphism order on the class of linegraphs of graphs with a vertices of degree at most d. We were able to give an affirmative answer.

Theorem 4.2 ([3]) *The homomorphism order of line graphs of regular graphs with maximal degree d is universal for every $d \geq 3$.*

This result may seem counter-intuitive with respect to the Vizing theorem. Vizing class 1 contains the graphs whose chromatic index is the same as the maximal degree of a vertex, while Vizing class 2 contains the remaining graphs. Because the Vizing class 1 is trivial it may seem that the homomorphism order on the Vizing class 2 should be simple, too. The converse is true.

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