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Triangle-free subgraphs with large fractional chromatic number

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Abstract

It is well known that for any k and g, there is a graph with chromatic number at least k and girth at least g. In 1970's, Erdős and Hajnal conjectured that for any numbers k and g, there exists a number $f(k, g)$, such that every graph with chromatic number at least $f(k, g)$ contains a subgraph with chromatic number at least k and girth at least g. In 1978, Rödl proved the case for $g = 4$ and arbitrary k. We prove the fractional chromatic number version of Rödl's result.

Keywords: Chromatic number, fractional chromatic number.

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1 Outline

A well known result, proved by Erdős in 1950s, tells us that for every k and g, there exists a graph with chromatic number at least k and girth at least g. In 1970s, Erdős and Hajnal conjectured that for any k and g, there exists an integer $f(k,g)$, such that every graph with chromatic number at least $f(k,g)$ contains a subgraph with chromatic number at least k and girth at least g. In 1977, Rödl [2] proved the conjecture for $g = 4$ and arbitrary k. The special case when $g = 4$ speaks about triangle-free subgraphs. This is the only nontrivial case for which the Erdős and Hajnal conjecture has been confirmed.

Theorem 1.1 (Rödl (1977)) *For every positive integer k, there exists an integer $f(k)$ such that if $\chi(G) \geq f(k)$ then G contains a triangle-free subgraph H with $\chi(H) = k$.*

Let $\mathcal{I}(G)$ be the family of all independent sets of G, and let $\mathcal{I}(G,v)$ be the family of all those independent sets which contain the vertex v. For each independent set I, consider a nonnegative real variable x_I. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the minimum value of

$$\sum_{I \in \mathcal{I}(G)} x_I, \quad \text{subject to} \sum_{I \in \mathcal{I}(G,v)} x_I \geq 1 \text{ for each } v \in V(G).$$

Erdős actually proved the existence of a graph with large girth and large fractional chromatic number, instead of chromatic number. This is a stronger statement as the chromatic number is always greater or equal to the fractional chromatic number. In this paper, we prove the fractional chromatic number version of Rödl's result.

Theorem 1.2 *For every positive integer k, there exists an integer $g(k)$ such that every graph G with $\chi_f(G) \geq g(k)$ contains a triangle-free subgraph H with $\chi_f(H) \geq k$.*

Given this result, we put forward the following.

Conjecture 1.3 *For every positive integers k and l, there exists an integer $g(k,l)$ such that every graph G with $\chi_f(G) \geq g(k,l)$ contains a subgraph H of girth at least l and with $\chi_f(H) \geq k$.*

As our final contribution, we show in Section 3 that the Erdős-Hajnal Conjecture holds for Kneser graphs.

2 Proof of Theorem 1.2

The proof of Theorem 1.2 uses the tools presented in this section.

Given an arbitrary ordering $v_1, \ldots, v_n$ of the vertices of a graph G, let

$$N^L(v_i) = \{v_j : v_j v_i \in E(G), j < i\},$$

that is, the set of the neighbors of v_i that appear before v_i. Rödl's proof of Theorem 1.1 is based on the following lemma.

Lemma 2.1 *If $\chi(G) > k^t$, and $\chi(N^L(v)) \le t$ for every $v \in V(G)$, then there exists a triangle-free subgraph H with $\chi(H) > k$.*

The original proof of Lemma 2.1 is elegant and very short, but it cannot be applied to the fractional chromatic number. Our major effort is to extend the above claim to the fractional chromatic number setup.

For a function $w : V(G) \to \mathbb{R}$, and a vertex set A, we write $w(A) = \sum_{v \in A} w(v)$. The *fractional independence number*, denoted by $\alpha_f(G)$, is the minimum over all non-negative weight functions w with $w(V) = n$, of the maximum value of $w(I)$ over all independent sets I. Via LP duality we have the fact that $\chi_f(G) = \frac{n}{\alpha_f(G)}$, where $n = |V(G)|$. So we can consider our problem as a fractional independence number problem.

The following is our statement on the fractional independence number analogous to Lemma 2.1. It uses the following function

$$f(k, l) = \left(\frac{(kl^7)!}{k!} \right)^3$$

defined for all positive integers k and l. We also fix a weight function $w : V(G) \to \mathbb{R}_+$ on the vertices of G. In the proof of Theorem 1.2, the function w is the one giving the fractional independence number of G.

Lemma 2.2 *Suppose that $w(I) \le \frac{w(V)}{f(x,l)}$ for every $I \in \mathcal{I}(G)$, and that $\chi_f(N^L(v)) \le l$ for every $v \in V(G)$. Then G contains a triangle-free subgraph H, such that $w(I) \le \frac{w(V(H))}{x}$ for every $I \in \mathcal{I}(H)$. In particular, $\chi_f(H) \ge x$.*

Let $v_1, \ldots, v_n$ be the enumeration of the vertices in the non-decreasing order according to the weight function w. Given a vertex set A, let $A_k = \{v_1, \ldots, v_k\}$ be the first k elements in A according to the ordering $v_1, \ldots, v_n$. Let A^k be the k-th element of A. A subset B of A is called *k-principal* in A if $B \subseteq A_{k|B|}$. A subset of A is called *k-sparse* in A if it contains no k-principal subset of A.

Let A be a vertex set and $v \in A$, let $L_A(v)$ be the graph induced by the neighbors of v in A that appear before v. The set A is *reducible* if it satisfies the following conditions:

(i) $w(A) \geq \frac{w(V)}{(x+1)^3}$ and

(ii) $\chi_f(L_A(v)) \leq l(1 - \frac{1}{6(x+1)})$.

Lemma 2.3 *For any reducible set A, there is a triangle-free subgraph H such that any independent set of H has weight at most $\frac{w(A)}{x+1}$.*

Proof. (Sketch) Note that $f(k, l)$ satisfies

$$f(k, l) \geq (k+1)^3 f(k+1, l(1 - \tfrac{1}{6(k+1)}))$$

and $f(k, 1) = 1$.

For any independent set $I \subseteq A$, we have

$$w(I) \leq \frac{w(V)}{f(x, l)} \leq \frac{w(A)}{\frac{f(x,l)}{(x+1)^3}} \leq \frac{w(A)}{f(x+1, l(1 - \frac{1}{6(x+1)}))}.$$

By using induction on l, there is a triangle-free subgraph such that any independent set has weight at most $\frac{w(A)}{x+1}$. □

Let R be the union of a maximal collection of disjoint reducible sets. Then the complement $\bar{R} = V \setminus R$ contains no reducible subsets. Applying Lemma 2.3, we can find a triangle-free subgraph of R, such that for any independent set $I \subset R$, we have $w(I) \leq \frac{w(R)}{x+1}$.

Let $L_G(v)$ denote the graph $L_{V(G)}(v)$. Assuming that $\chi_f(L_G(v)) \leq l$, let $\mathcal{I}(v)$ be the collection of independent set of $L_G(v)$. There exists a weight function $u : \mathcal{I}(v) \to [0, 1]$, such that any vertex in $L_G(v)$ is covered by independents with total weight at least 1, and total weight of $\mathcal{I}(v)$ equals l. For a set A containing v, we say v is *type* 1 in A if the total weight of independent sets in $\mathcal{I}(v)$ that is totally out of A is at most $\frac{l}{6(x+1)}$; otherwise v is *type* 2 in A. Let $T_1(A)$ be the collection of type 1 vertices of A and $T_2(A)$ be the collection of type 2 vertices of A.

Given a vertex set $A \subseteq \bar{R}$, we say A is *dense* if A is $(x+1)$-principal in $\bar{R}$ and $|T_2(A)| \leq \frac{|A|}{x+1}$.

Lemma 2.4 *There is a triangle-free subgraph H, such that if A is dense, then A is not stable in H.*

Sketch of the proof: Let H be a subgraph of $\bar{R}$, obtained randomly using the following random choice. For each vertex $v \in R$, randomly pick an independent I from $\mathcal{I}(v)$ according to their weight u, and then add the edges between I and v to H. Now, the lemma follows from the following claims that are not too hard to prove.

Claim 1 *H is a triangle-free subgraph of G.*

Claim 2 *The probability that every dense subset of $\bar{R}$ contains an edge in H is positive.*

Lemma 2.5 *If a set $S \subseteq \bar{R}$ contains no dense subset, then $w(S) \leq \frac{w(\bar{R})}{x+1} + \frac{w(V)}{(x+1)^2}$.*

Claim 3 *Every independent set I has weight at most $\frac{w(V)}{x}$.*

Proof. $w(I) = w(I \cap R) + w(I - R) \leq \frac{w(R)}{x+1} + \frac{w(\bar{R})}{x+1} + \frac{w(V)}{(x+1)^2} = w(V)(\frac{1}{x+1} + \frac{1}{(x+1)^2}) \leq \frac{w(V)}{x}$. □

3 Blow-ups and Kneser graphs

This section contains the proof that the Erdős-Hajnal Conjecture holds for Kneser graphs.

Given a graph H, the *blow-up* of H with *power* m, denoted by $H^{(m)}$, is the graph obtained from H by replacing each vertex by an independent set of size m (called the *blow-up* of the vertex), and for each edge xy in H, the two blow-ups of x and y form a complete bipartite graph $K_{m,m}$. The subgraph of $H^{(m)}$ replacing an edge xy of H is isomorphic to $K_{m,m}$ and will be referred to as the *blow-up* of that edge.

We have the following statement.

Theorem 3.1 *Suppose G is a graph with $\Delta(G) \leq \Delta$ and $\chi(G) > x$. Suppose that m is an integer that is larger than $x(x\Delta)^{2g-4}$. Then there exists a subgraph H of $G^{(m)}$ with girth more than g and chromatic number more than x.*

There are several existing papers, for example [1] or [3], that gave a result similar to Theorem 3.1, which are used to prove the existence of uniquely colorable graphs. But the bound for the blow-up power m in those papers is too large for our purpose as it depends on the number of vertices of G instead of the maximum degree.

To prove Theorem 3.1, we use the following fact.

Claim 4 *Given G with $\chi(G) > x$, let H be a subgraph of $G^{(m)}$. Suppose that for any edge $ab \in E(G)$ and for any subsets X, Y contained in the blow-ups of a and b, respectively, with $|X| \geq \frac{m}{x}$, $|Y| \geq \frac{m}{x}$, there is an edge between X and Y in H. Then $\chi(H) > x$.*

Consider a random subgraph H of $G^{(m)}$, obtained in such a way that each edge is picked with probability $(\frac{m}{x})^{\frac{1}{4l}-1}$. Consider the event that the subgraph has no short cycle, and for any edge $ab \in E(G)$ and for any pair X, Y contained in the respective blow-ups of a and b, and with $|X| \geq \frac{m}{x}$, $|Y| \geq \frac{m}{x}$, there is an edge between X and Y in H. Applying the Asymmetric Form of the Lovász Local Lemma to the two types of events together by carefully choosen parameters, we are able to show that the event has positive probability.

From Theorem 3.1, we know that graphs that are blow-ups of smaller graphs with sufficiently large power satisfy the Erdős-Hajnal Conjecture. In particular, Kneser graphs are such examples. This is certified by the following theorem (which is proved in the full version of the paper).

Theorem 3.2 *Let n, k, t, and x be nonnegative integers such that $0 < k < n$ and $x < kt$. The Kneser graph $KG(nt, kt - x)$ contains the blow-up of $KG(n, k)$ with power $\binom{k(t-1)}{x}$ as a subgraph. Furthermore, when $x < t$, it contains the blow-up of $KG(n, k)$ with power $\binom{kt}{x}$, and when $x = t$, it contains the blow-up of $KG(n, k)$ with power $\binom{kt}{x} - k$.*

The results listed above can be used to derive the main result of this section.

Corollary 3.3 *The Erdős-Hajnal Conjecture holds for Kneser graphs.*

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