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On the Minimum Edge-Density of 5-Critical Triangle-Free Graphs

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Abstract

Kostochka and Yancey proved that every 5-critical graph G satisfies: $|E(G)| \geq \frac{9}{4}|V(G)| - \frac{5}{4}$. A construction of Ore gives an infinite family of graphs meeting this bound.

We prove that there exists $\epsilon, \delta > 0$ such that if G is a 5-critical graph, then $|E(G)| \geq (\frac{9}{4} + \epsilon)|V(G)| - \frac{5}{4} - \delta T(G)$ where $T(G)$ is the maximum number of vertex-disjoint cliques of size three or four where cliques of size four have twice the weight of a clique of size three. As a corollary, a triangle-free 5-critical graph G satisfies: $|E(G)| \geq (\frac{9}{4} + \epsilon)|V(G)| - \frac{5}{4}$.

Keywords: Graph Coloring, Critical Graphs, Triangle-Free Graphs, Edge Density

1 Introduction

Intuitively, a graph that has fewer edges can be properly colored by a smaller number of colors. Kostochka and Yancey [2] confirmed this intuition recently by proving that every non-4-colorable graph has a subgraph with large edge

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density. We say that a graph is 5-*critical* if it is not 4-colorable, but all of its proper subgraphs are.

Theorem 1.1 (Kostochka, Yancey [2]) *If G is a 5-critical graph on n vertices, then*

$$|E(G)| \geq \frac{9n-5}{4}.$$

Furthermore, the bound in Theorem 1.1 is tight since it is attained by infinitely many 5-critical graphs. In fact, in a subsequent paper [3], Kostochka and Yancey characterized the 5-critical graphs that attain these bounds. First a definition.

Definition 1.2 An *Ore-composition* of graphs G_1 and G_2 is a graph obtained by the following procedure:

(i) delete an edge xy from G_1;

(ii) split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree;

(iii) identify x with z_1 and identify y with z_2.

We say that G_1 is the *edge-side* and G_2 the *split-side* of the composition. Furthermore, we say that xy is the *replaced edge* of G_1 and that z is the *split vertex* of G_2. We say that G is a *k-Ore graph* if it can be obtained from copies of K_k and repeated Ore-compositions.

Theorem 1.3 *If G be a 5-critical, then $|E(G)| = \frac{9n-5}{4}$ if and only if G is a* 5-Ore *graph.*

More generally, answering a question of Gallai and Ore, Kostochka and Yancey [2] showed that if G is a k-critical graph, then $|E(G)| \geq (\frac{k}{2} - \frac{1}{k-1})|V(G)| - \frac{k(k-3)}{2(k-1)}$, which is tight for k-Ore graphs. In a previous work [4], the author showed that this can be improved for 4-critical graphs of girth five as follows: There exists $\epsilon > 0$ such that if G is a 4-critical graph of girth at least five, then $|E(G)| \geq (\frac{5}{3} + \epsilon)|V(G)| - \frac{2}{3}$. It is natural then to wonder if a similar results holds for larger k. Our main result answers this in the affirmative for 5-critical triangle-free graphs but first a definition.

Definition 1.4 If H is a disjoint union of cliques of size three or four, then we let $T(H)$ be the number of components in H that are cliques of size three plus twice the number of components which are cliques of size four. More generally, we let $T(G)$ denote the maximum of $T(H)$ over all such subgraph H of G.

Here is our main result.

Theorem 1.5 *There exists $\delta, \epsilon, P > 0$ such that the following holds. Let $p(G) = (9+\epsilon)|V(G)| - 4|E(G)| - \delta T(G)$. If G is a 5-critical graph, then*

(i) $p(G) = 5 + 5\epsilon - 2\delta$ *if* $G = K_5$,

(ii) $p(G) \leq 5 + |V(G)|\epsilon - (2 + \frac{(|V(G)|-1)}{4})\delta$ *if G is 5-Ore and* $G \neq K_5$,

(iii) $p(G) \leq 5 - P$ *otherwise.*

Corollary 1.6 *There exists $\epsilon > 0$ such that if G is a 5-critical triangle-free graph, then $|E(G)| \geq (\frac{9}{4} + \epsilon)|V(G)| - \frac{5}{4}$.*

Note that $T(K_5) = 2$. Hence $p(K_5) = 5 + 5\epsilon - 2\delta$.

2 Outline of Proof

To prove the second assertion of Theorem 1.5, we prove the following lemma:

Lemma 2.1 *If $G \neq K_5$ is a 5-Ore graph, then $T(G) \geq 2 + \frac{|V(G)|-1}{4}$.*

Note the following observation.

Lemma 2.2 *If G is the Ore-composition of two graphs G_1 and G_2, then $T(G) \geq T(G_1) + T(G_2) - 2$. Furthermore if $G_2 = K_4$, then $T(G) \geq T(G_1) + 1$.*

Proof. To prove the first statement, without loss of generality let e be the replaced edge of G_1 and z the split vertex of G_2. It follows that $T(G) \geq T(G_1 - e) + T(G_2 \setminus z)$. But $T(G_1) - e \geq T(G_1) - 1$ and $T(G_2 \setminus z) \geq T(G_2) - 1$. Hence $T(G) \geq T(G_1) + T(G_2) - 2$ as desired. To prove the second statement, note that for every edge $e \in E(K_5)$, $T(K_5 - e) = 2$ and for every vertex $z \in V(K_5)$, $T(K_5 \setminus z) = 2$. Thus in either case, it follows from the calculations above that $T(G) \geq T(G_1) - 1 + 2 = T(G_1) + 1$. □

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. We proceed by induction on $|V(G)|$. Since $G \neq K_5$ and G is 5-Ore, G is the Ore-composition of two graphs G_1 and G_2. For each $i \in \{1, 2\}$, if $G_i \neq K_5$, then by induction $T(G_i) \geq 2 + \frac{|V(G_i)|-1}{4}$.

First suppose that neither G_1 nor G_2 is isomorphic to K_5. By Lemma 2.2, $T(G) \geq T(G_1) + T(G_2) - 2$. Using induction, we find that $T(G) \geq 2 + \frac{|V(G_1)|-1}{4} + 2 + \frac{|V(G_2)|-1}{4} - 2 = 2 + \frac{|V(G_1)|+|V(G_2)|-2}{4}$. Yet $|V(G)| = |V(G_1)| + |V(G_2)| - 1$. So $T(G) \geq 2 + \frac{|V(G)|-1}{4}$ as desired.

So we may assume without loss of generality that $G_2 = K_5$. Next suppose $G_1 \neq K_5$. By Lemma 2.2, $T(G) \geq T(G_1) + 1$. By induction, $T(G_1) \geq$

$2 + \frac{|V(G_1)|-1}{4}$. So $T(G) \geq 3 + \frac{|V(G_1)|-1}{4}$. Yet $|V(G)| = |V(G_1)| + 4$, so $T(G) \geq 2 + \frac{|V(G)|-1}{4}$ as desired.

Finally suppose both G_1 and G_2 are isomorphic to K_5. Without loss of generality, let e be the replaced edge of G_1 and z the split vertex of G_2. Then $T(G_1 - e) = T(G_2 \setminus z) = 2$ as both contain a K_4. Hence $T(G) \geq T(G_1 - e) + T(G_2 \setminus z) = 2 + 2 = 4$. Meanwhile, $|V(G)| = 5 + 5 - 1 = 9$. Thus, $T(G) = 4 \geq 2 + \frac{9-1}{4}$ as desired. $\square$

2.1 Potential

The proof of the third assertion of Theorem 1.5 is quite long and technical. Thus we only provide an outline of the proof. The proof is modeled on Kostochka and Yancey's proof of Theorem 1.1. They defined a potential for 5-critical graphs, let us call it the *Kostochka-Yancey potential* of a graph G , denoted $p_{KY}(G)$ as $9|V(G)| - 4|E(G)|$. Theorem 1.1 then says that every 5-critical graph satisfies $p_{KY}(G) \leq 5$ while Theorem 1.3 says that equality holds if and only if G is 5-Ore.

Definition 2.3 If $R \subsetneq V(G)$ with $|R| \geq 5$, and ϕ is a 4-coloring of $G[R]$, we define the *ϕ-identification of R in G*, denoted by $G_\phi(R)$, to be the graph obtained from G by identifying for each $i \in \{1,2,3,4\}$ the vertices colored i in R to a vertex x_i, adding the edges x_ix_j for all $i,j \in \{1,2,3,4\}$ and then deleting parallel edges.

Proposition 2.4 *If G is 5-critical, $R \subsetneq V(G)$ with $|R| \geq 5$, and ϕ is a 4-coloring of $G[R]$, then $\chi(G_\phi(R)) \geq 5$.*

Since the resulting graph contains a 5-critical graph, we may extend the set R to a larger set as follows:

Definition 2.5 Let G be a 5-critical graph, $R \subsetneq V(G)$ with $|R| \geq 5$ and ϕ a 4-coloring of $G[R]$. Now let W be a 5-critical subgraph of $G_\phi(R)$ and X be the graph on the set of vertices x_i (Note not all such vertices may exist). Then we say that $R' = (V(W) - V(X)) \cup R$ is the *critical extension* of R with *extender* W. We call $W \cap X$ the *core* of the extension.

Note that every critical extension has a non-empty core as otherwise G would contain a proper non-4-colorable subgraph contradicting that G is 5-critical. Here is a key lemma bounding our potential for critical extensions in terms of the original set and the extending critical graph. Note the use of the vertex-disjointness of $T(G)$.

Lemma 2.6 *For small enough δ and ϵ the following holds: If G is a 5-critical graph, $R \subsetneq V(G)$ with $|R| \geq 5$ and R' is a critical extension of R with extender W and core X, then*

$$p_G(R') \leq p_G(R) + p(W) - f(|X|) + \delta(T(W) - T(W \setminus X)),$$

where $f(|X|) = p(K_{|X|}) - T(X)$.

Furthermore,

$$p_G(R') \leq p_G(R) + p(W) - 9 - \epsilon + \delta.$$

2.2 Properties of a Minimum Counterexample

Let G be a minimum counterexample. Here are two key structural lemmas that are proved about a minimum counterexample. Their proofs are intricate and so are omitted. Furthermore, they require some machinery about the potential of extensions and the characterizations of certain reductions which are also omitted for brevity. Before we state these two lemmas, first we need a definition.

Definition 2.7 We define $D_4(G)$ to be the subgraph of G induced by the vertices of degree four. A graph is *almost 5-Ore* if it can be obtained from a 5-Ore graph by deleting a vertex in a cluster of size at least two.

Lemma 2.8 *If uv is an edge of $D_4(G)$, then u is contained in a subgraph of G not containing v that is almost 5-Ore. Furthermore, every component of $D_4(G)$ has size at most 2.*

Lemma 2.9 *If v is a vertex of degree 5 in G, then v has at most one neighbor of degree four that is incident with an edge of $D_4(G)$.*

2.3 Discharging

We now outline the discharging proof of Theorem 1.5. Let G be a minimum counterexample as in the previous section. We will need the following theorem of Kierstead and Rabern [1]:

Definition 2.10 The *maximum independent count* of a graph G, denoted $\mathrm{mic}(G)$, is the maximum of $\sum_{v \in I} d(v)$ over all independent sets I of G.

Theorem 2.11 *If G is a k-critical graph, then*

$$|E(G)| \geq \frac{k-2}{2}|V(G)| + \frac{1}{2}\text{mic}(G).$$

We proceed by discharging. Let the charge of a vertex v, denoted $ch(v)$ be given by $ch(v) = (9+\epsilon) - 2d(v)$. We now discharge according to the following rule to obtain a new charge, denoted $ch_F(v)$.

Discharging Rule: If v is a vertex of degree at least 5 with a neighbor u of degree four in a componenet of $D_4(G)$ of size at least two, then v receives $+1/3$ charge from u.

Lemma 2.12 *If v has degree at least 5, then $ch_F(v) \leq -2/3 + \epsilon$.*

Proof. If v has degree 5, then $ch(v) = -1+\epsilon$. By Lemma 2.9, v sends charge to at most one neighbor. Hence $ch_F(v) \geq -1 + \epsilon + 1/3 = \epsilon - 2/3$ as desired.

Suppose then that v has degree at least 6. Now, $ch(v) = (9+\epsilon) - 2d(v)$ and v receives at most $+1/3$ charge from each neighbor. Hence $ch_F(v) \leq (9+\epsilon) - 2d(v) + d(v)/3 = 9 + \epsilon - \frac{5}{3}d(v)$. As $d(v) \geq 6$, this is at most $-1+\epsilon$ as desired. □

However, if v has degree four and is in a component of size two of $D_4(G)$, then $ch_F(v) = \epsilon$. Meanwhile if v is degree four and in a component of size 1 of $D_4(G)$, then $ch_F(v) = 1 + \epsilon$. Let S be the number of components of size one in $D_4(G)$ and M be the number of components of size two in $D_4(G)$. Hence the number of vertices of degree four is $S + 2M$ and the number of vertices of degree at least five is $|V(G)| - S - 2M$. Note that there is an independent set consisting of vertices of degree four of size at least $S + M$. Hence $\text{mic}(G) \geq 4(S+M)$.

Thus $|E(G)| \geq \frac{3}{2}|V(G)| + 2(S+M)$. As $p(G) > 0$, $(9+\epsilon)|V(G)| > 4|E(G)|$. Hence $(9+\epsilon)|V(G)| > 6|V(G)| + 8(S+M)$. Thus $S + M < \frac{3+\epsilon}{8}|V(G)|$. On the other hand, $\sum_v ch(v) = (9+\epsilon)|V(G)| - 2\sum_v d(v) = (9+\epsilon)|V(G)| - 4|E(G)| \geq p(G) > 0$. Hence $\sum_v ch_F(v) > 0$. Yet, $\sum_v ch_F(v) \leq -\frac{2}{3}(|V(G)| - S - 2M) + S + \epsilon|V(G)|$. Thus, $\frac{5}{3}S + \frac{4}{3}M > (\frac{2}{3} - \epsilon)|V(G)|$.

So on the one hand, $S + M > \frac{3}{5}(\frac{2}{3} - \epsilon)|V(G)|$ and, on the other hand, $S + M < \frac{3+\epsilon}{8}|V(G)|$. That is, $\frac{3+\epsilon}{8}|V(G)| > \frac{2-3\epsilon}{5}|V(G)|$. Hence, $15 + 5\epsilon > 16 - 24\epsilon$. That is, $29\epsilon > 1$. So $\epsilon > \frac{1}{29}$, a contradiction.

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