Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Many T copies in H-free graphs

Noga Alon [1,2]

Sackler School of Mathematics and Blavatnik School of Computer Science
Tel Aviv University
Tel Aviv, Israel

Clara Shikhelman [3]

Sackler School of Mathematics
Tel Aviv University
Tel Aviv, Israel

Abstract

For two graphs T and H with no isolated vertices and for an integer n, let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H-free graph on n vertices. The study of this function when $T = K_2$ is a single edge is the main subject of extremal graph theory. In the present paper we investigate the general function, focusing on the cases of triangles, complete graphs, complete bipartite graphs and trees. These cases reveal several interesting phenomena. Three representative results are:

(i) $ex(n, K_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2}n^{3/2}$

(ii) For any fixed m, $s \geq 2m-2$ and $t \geq (s-1)!+1$, $ex(n, K_m, K_{s,t}) = \Theta(n^{m-\binom{m}{2}/s})$

(iii) For any two trees H and T one has $ex(n, T, H) = \Theta(n^m)$ where $m = m(T, H)$ is an integer depending on H and T (its precise definition is given in the introduction).

The first result improves (slightly) an estimate of Bollobás and Győri. The proofs combine combinatorial and probabilistic arguments with simple spectral techniques.

Keywords: Extremal Combinatorics, Turán-type problems.

1 Introduction

For two graphs T and H and for an integer n, let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H-free graph on n vertices.

When $T = K_2$ is a single edge, $ex(n, T, H)$ is the well studied function, usually denoted by $ex(n, H)$, specifying the maximum possible number of edges in an H-free graph on n vertices. There is a huge literature investigating this function, starting with the theorems of Mantel [12] and Turán [16] that determine it for $H = K_r$. See, for example, [14] for a survey.

In the present abstract we show that the function for other graphs T besides K_2 exhibits several additional interesting features. We illustrate these by focusing on a few special cases such as the triangle $T = K_3$ a general complete graph $T = K_m$ or a tree, but the question is interesting for many other graphs T, and many of the results can be extended to other graphs.

There are several sporadic papers dealing with the function $ex(n, T, H)$ for $T \neq K_2$. The first one is due to Erdős in [6], where he determines $ex(n, K_t, K_r)$ for all $t < r$ (see also [4] for an extension). A notable recent example is given in [9], where the authors determine this function precisely for $T = C_5$ and $H = K_3$. Another example is $T = K_r$ and $H = K_t$ where $r < t$, which follows from the results in [4].

The case $T = K_3$ and $H = C_{2k+1}$ has also been studied. Bollobás and Győri [5] proved that

$$(1 + o(1))\frac{1}{3\sqrt{3}}n^{3/2} \leq ex(n, K_3, C_5) \leq (1 + o(1))\frac{5}{4}n^{3/2}. \tag{1}$$

Győri and Li [8] proved that for any fixed $k \geq 2$

$$\binom{k}{2} ex_{bip}(\frac{2n}{k+1}, C_4, C_6, \ldots, C_{2k}) \leq ex(n, K_3, C_{2k+1}) \leq \frac{(2k-1)(16k-2)}{3} ex(n, C_{2k}), \tag{2}$$

[1] Sackler School of Mathematics and Blavatnik School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel and School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540. Research supported in part by a USA-Israeli BSF grant, by an ISF grant, by the Israeli I-Core program and by the Oswald Veblen Fund.
[2] Email: nogaa@tau.ac.il
[3] Email: clarashk@mail.tau.ac.il

Our first result characterizes all graphs H for which $ex(n, K_3, H) \leq c(H)n$. The friendship graph F_k is the graph consisting of k triangles with a common vertex. Call a graph an extended friendship graph iff its 2-core is either empty or F_k for some positive k.

Theorem 1.1 *There exists a constant $c(H)$ so that $ex(n, K_3, H) \leq c(H)n$ if and only if H is a subgraph of an extended friendship graph.*

We also slightly improve the upper estimates in (1) and in (2) above, proving the following.

Proposition 1.2 *The following upper bounds hold.*
(i) $ex(n, K_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2}n^{3/2}$.
(ii) For any $k \geq 2$, $ex(n, K_3, C_{2k+1}) \leq \frac{16(k-1)}{3}\ ex(\lceil n/2 \rceil, C_{2k})$.

A similar result has been proved independently by Füredi and Özkahya [7], who showed that $ex(n, K_3, C_{2k+1}) \leq 9k\ ex(n, C_{2k})$.

The next theorem deals with maximizing the number of copies of a complete graph while avoiding bipartite graphs:

Theorem 1.3 *For any fixed m and $t \geq s$ satisfying $s \geq 2m-2$ and $t \geq (s-1)! + 1$ there are two constants $c_1 = c_1(s,t)$ and $c_2 = c_2(s,t)$ such that*

$$c_1 n^{m-\binom{m}{2}/s} \leq ex(n, K_m, K_{s,t}) \leq c_2 n^{m-\binom{m}{2}/s}.$$

The last two theorem focus on the case where H is a tree. Before we state the results let us give the following definitions:

Definition 1.4 For a graph T, a set of vertices $U \subseteq V(T)$ and an integer h, the (U, h) blow-up of T is the following graph. Fix the vertices in U, and replace each connected component in $T \setminus U$ with h vertex disjoint copies of it connected to the vertices of U exactly as the original component is connected to these in T.

Definition 1.5 For two trees, T and H, let $m(T, H)$ be the maximum integer m such that there is a $(U, |V(H)|)$ blow-up of T containing no copy of H and having m connected components in $T \setminus U$.

In this notation we prove the following.

Theorem 1.6 *Let H and T be trees and $m = m(T, H)$. Then there are two positive constants $c_1(t, h), c_2(t, h)$ so that*

$$c_1(t, h)n^m \leq ex(n, T, H) \leq c_2(t, h)n^m$$

Finally we consider the case where T is a bipartite graph and H is a tree. For a tree H, any H-free graph can have at most a linear number of edges. Therefore, by a theorem proved in [1], the maximum possible number of copies of any bipartite graph T in it is bounded by $O(n^{\alpha(T)})$, where $\alpha(T)$ is the size of a maximum independent set in T. Using the next definition we characterize the cases in which $ex(n, T, H) = \Theta(n^{\alpha(T)})$.

Definition 1.7 An edge cover of a graph T is a set $\Gamma \subset E(T)$ such that for each vertex $v \in V(T)$ there is an edge $e \in \Gamma$ for which $v \in e$. Call an edge-cover minimum if it has the smallest possible number of edges.

A set of vertices $U \subset V(T)$ is called a $U(\Gamma)$-set if each connected component of $T \setminus U$ intersects exactly one edge of Γ, and the number of these connected components is $|\Gamma|$.

Theorem 1.8 *Let T be a bipartite graph and let H be a tree. Then the following are equivalent:*

(i) $ex(n, T, H) = \Theta(n^{\alpha(T)})$

(ii) *For any minimum edge-cover Γ of T there is a choice of a $U(\Gamma)$-set U such that the (U, h) blow-up of T does not contain a copy of H,*

(iii) *For some minimum edge cover Γ of T there is a choice of a $U(\Gamma)$-set U such that the (U, h) blow-up of T does not contain a copy of H.*

2 Some Proof Sketches

2.1 Proof of Theorem 1.1

For the proof of Theorem 1.1 we use the following two lemmas:

Lemma 2.1 *Let $G = (V, E)$ be a graph with at least $(9c-15)(c+1)n$ triangles and at most n vertices, then it contains a copy of F_c.*

Lemma 2.2 *For every $k > 3$ and n large enough there is a graph G on n vertices with at least $\Omega(n^{1+\frac{1}{k-1}})$ triangles and no cycles of length i for any i between 4 and k.*

We can now prove Theorem 1.1.

Proof. (**Sketch**) We need to show first that $ex(n, K_3, H)$ is linear in n for any extended friendship graph and second that if H is not a subgraph of an extended friendship graph then there is a graph G with n vertices and $\omega(n)$ triangles containing no copy of H.

The first part can be achieved by showing that a graph with at least $10h^2 \cdot n$ triangles has a subgraph with many triangles and a high minimal degree. By Lemma 2.1 this graph will contains a copy of the 2-core of H and the high minimal degree allows us to embed all of H.

For the second part we note that H is not a subgraph of an extended friendship graph iff it either contains a cycle of length greater than 3 or it contains two vertex disjoint triangles. Lemma 2.2 provides a graph G with a superlinear number of triangles and no no copy of C_k for $k \geq 4$. The complete 3-partite graph $K_{1,\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ has a non-linear number of triangles and non of them are disjoint. □

2.2 *Proof of Theorem 1.3*

We prove the upper and lower bound in the following two lemmas. From here on, denote by $\mathcal{N}(G, H)$ the number of copies of H in G.

Lemma 2.3 *For any fixed $m \geq 2$ and $t \geq s \geq m-1$*

$$ex(n, K_m, K_{s,t}) \leq (\frac{1}{m!} + o(1))(t-1)^{\frac{m(m-1)}{2s}} n^{m - \frac{m(m-1)}{2s}}$$

Proof. (**Sketch**) We apply induction on m. For $m = 2$ the Kövari, Sós Turán result [11] gives $ex(n, K_2, K_{s,t}) = ex(n, K_{s,t}) \leq (\frac{1}{2} + o(1))(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}}$ and this will serve as our base case.

For the induction step assume we have proved this for m and let us prove it for $m+1$. Let $G = (V, E)$ be a $K_{s,t}$ free graph on n vertices, and let us bound the number of copies of K_{m+1} in it. For each $v \in V$ we know that its neighborhood $N(v)$ does not contain any copy of $K_{s-1,t}$. By the induction assumption we can bound the number of copies of K_m in $N(v)$:

$$\mathcal{N}(N(v), K_m) \leq ex(d_v, K_m, K_{s-1,t}) \leq (\frac{1}{m!} + o(1))(t-1)^{\frac{m(m-1)}{2(s-1)}} d_v^{m - \frac{m(m-1)}{2(s-1)}}$$

From this and by using the means inequality and the fact that the number of s-edged stars in G cannot exceed $\binom{n}{s}(t-1)$ we can get a bound on the number of copies of K_m in G:

$$\mathcal{N}(G,K_{m+1}) \le \left(\frac{1}{(m+1)!}+o(1)\right)(t-1)^{\frac{(m+1)m}{2s}} n^{(m+1)-\frac{(m+1)m}{2s}}$$

□

Lemma 2.4 *For any fixed m, $s \ge 2m-2$ and $t \ge (s-1)!+1$*

$$ex(n,K_m,K_{s,t}) \ge \left(\frac{1}{m!}+o(1)\right)n^{m-\frac{m(m-1)}{2s}}$$

Proof. (**Sketch**)

We use the projective norm-graphs $H(q,s)$ as constructed in [3], in the same paper it is shown that $H(q,s)$ is $K_{s,(s-1)!+1}$ free. To determine the number of copies of K_m in this graph we use a result on (n,d,λ) graphs. An (n,d,λ) graph is a d-regular graph on n vertices in which all eigenvalues but the first have absolute value at most λ.

A result of the first author (see [10], Theorem 4.10) is the following: Let G_1 be a fixed graph with r edges, s vertices and maximum degree Δ. Let G_2 be an (n,d,λ) graph. If $n \gg \lambda(\frac{n}{d})^{\Delta}$ then the number of copies of G_1 in G_2 is $(1+o(1))\frac{n^s}{|Aut(G_1)|}(\frac{d}{n})^r$. In our case we take $G_1 = K_m$ and $G_2 = H(q,s)$. By the results in [15] or [2] we know that the second eigenvalue, in absolute value, of $H(q,s)$ is $q^{\frac{s-1}{2}}$ and from the construction $|V(H)| = q^s - q^{s-1}$, thus: $\mathcal{N}(H(q,s),K_m) = (\frac{1}{m!}+o(1))n^{m-\frac{m(m-1)}{2s}}$

□

References

[1] N. Alon, On the number of subgraphs of prescribed type of graphs with a given number of edges, Israel J. Math. 38(1981), 116-130.

[2] N. Alon and V. Rödl, Sharp bounds for some multicolor Ramsey numbers, Combinatorica 25 (2005), 125-141.

[3] N. Alon, L. Rónyai and T. Szabó, Norm-graphs: variations and applications, J. Combinatorial Theory, Ser. B 76 (1999), 280-290.

[4] B. Bollobás, On complete subgraphs of different orders, Math. Proc. Cambridge Philos. Soc. 79 (1976), 19-24.

[5] B. Bollobás and E. Győri, Pentagons vs. triangles, Discrete Math. 308 (2008), 4332–4336

[6] P. Erdős, On the number of complete subgraphs contained in certain graphs. Magyar Tud. Akad. Mat. Kut. Int. Kőzl, 7 (1962), 459–474.

[7] Z. Füredi and L. Özkahya, On the maximum number of triangles in special families of graphs and hypergraphs, in preparation.

[8] E. Győri and H. Li, The maximum number of triangles in C_{2k+1}-free graphs, Combinatorics, Probability and Computing 21(1-2), 187-191 (2012).

[9] H. Hatami, J. Hladký, D. Král', S. Norine and A. Razborov, On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A 120 (2013), no. 3, 722–732.

[10] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in: More sets, graphs and numbers, E. Gyori, G. O. H. Katona and L. Lovász, Eds., Bolyai Society Mathematical Studies Vol. 15, 199–262.

[11] T. Kövari, V. Sós, and P. Turán, On a problem of K. Zarankiewicz, Colloquium Math. 3 (1954), 50-57.

[12] W. Mantel, Problem 28, Wiskundige Opgaven 10 (1907), 60-61.

[13] I. Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, in Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18, Volume II, 939–945.

[14] M. Simonovits, Paul Erdős' influence on extremal graph theory, in: The mathematics of Paul Erdős, II, 148–192, Algorithms Combin., 14, Springer, Berlin, 1997.

[15] T. Szabó, On the spectrum of projective norm-graphs, Inform. Process. Lett. 86 (2003), no. 2, 71–74.

[16] P. Turán, On an extremal problem in graph theory (in Hungarian), Mat. Fiz. Lapok 48 (1941), 436-452.