Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Equitable colorings of non-uniform simple hypergraphs

Irina Shirgazina [1,2]

Department of Probability Theory, Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
Moscow, Russia

Abstract

The paper is devoted to the combinatorial problem concerning equitable colorings of non-uniform simple hypergraphs. Let $H = (V, E)$ be a hypergraph, a coloring with r colors of its vertex set V is called equitable if it is proper (i.e. none of the edges is monochromatic) and the cardinalities of the color classes differ by at most one. We show that if H is a simple hypergraph with minimum edge-cardinality n and

$$\sum_{e\in E} r^{1-|e|} \leqslant c\sqrt{n},$$

for some absolute constant $c > 0$, then H has an equitable r-coloring.

Keywords: Hypergraphs, colourings, Property B.

[1] This work was supported by Russian Foundation of Fundamental Research (grant 15-01-03530-) and by the grant of the President of Russian Federation MK-692.2014.1
[2] Email: ishirgazina@yandex.ru

1 Introduction

The paper deals with some aspects of the well-known problem of Erdős and Lovász concerning colorings of non-uniform hypergraphs.

Let us begin our review with well-known extremal problems concerning colorings of hypergraphs is the classical Property B problem stated by P. Erdős and A. Hajnal. The problem is to find the value $m(k)$ equal to the minimum possible number of edges in a k-uniform hypergraph which is not 2-colorable (see the survey [1] for the details). We shall recall a fragment from its history.

In 1973 Erdős and L. Lovász in their seminal paper [2] conjectured that

$$\frac{m(k)}{2^k} \to \infty \quad ask \to \infty.$$

Furthermore, they formulated a stronger conjecture concerning non-uniform hypergraphs. For a hypergraph $H = (V, E)$, let $f(H)$ denote the function

$$f(H) = \sum_{e \in E} 2^{-|e|}.$$

Erdős and Lovász proposed to consider the value $f(k)$ equal to the minimum possible value of $f(H)$ where H is 3-chromatic hypergraph with minimum edge-cardinality k. They conjectured that

$$f(k) \to \infty \quad ask \to \infty.$$

Both conjectures were proved by J. Beck in 1977–78. He proved that $m(k) \geqslant 2^k k^{1/3-o(1)}$, but his lower bound for $f(k)$ was much weaker. Using the function $\log^*(x)$ Beck's result (see [3]) can be formulated as follows:

(1) $$f(k) \geqslant \frac{\log^*(k) - 100}{7},$$

where $\log^*(k)$ is the inverse function for the tower of twos of height k. Thus this inequality proves the conjecture of Erdős and Lovász, but grows very slowly. In [4] D. Shabanov improved Beck's condition (which guarantees r-colorability in terms of $f(H)$) for simple triangle-free hypergraphs. He showed that if $H = (V, E)$ is triangle-free simple hypergraph with minimum edge-cardinality k and

(2) $$f_r(H) = \sum_{e \in E} r^{1-|e|} \leqslant \frac{1}{2} \left(\frac{k}{\ln k} \right)^{2/3},$$

then H is r-colorable.

In the current paper we investigate similar questions for equitable colorings of simple hypergraphs. The main result is formulated in the following theorem.

Theorem 1.1 *Suppose $r \geqslant 2$ and let $H = (V, E)$ be a simple hypergraph with minimum edge-cardinality k on n vertices, $r|n$. If*

$$f_r(H) = \sum_{e \in E} r^{1-|e|} \leqslant c \cdot \sqrt{k}, \tag{3}$$

for some absolute constant $c > 0$, then H has an equitable r-coloring.

Note that our condition for $f_r(H)$ is weaker than in shab, but we show the existence of a stronger equitable coloring instead of usual proper colorings. In the next section we shall give some key points of the proof of Theorem 1.1 using some ideas of random recoloring from the remarkable paper [6] of J. Radhakrishnan and A. Srinivasan.

2 Sketch of proof of Theorem 1.1

We have to show the existence of an equitable r-coloring for given hypergraph $H = (V, E)$. We shall construct some random coloring with equal sizes of color classes and prove that it will be proper with positive probability.

Let W be a vertex set of size n and let us split it into r subsets $W_1, \ldots, W_r$ of equal sizes n/r. We color the vertices of W with r colors: every $v \in W_i$ is colored with color i for all $i = 1, \ldots, r$. Let us define, for any $w \in W$, its "dual" vertices according with the following rules. If r is even, then we take an arbitrary fixed bijection $f_q : W_{2q-1} \to W_{2q}$, $q = 1, \ldots, r/2$ and define the bijection $T : W \to W$ as follows:

$$T(w) = \{ f_q(w), \text{ if } w \in W_{2q-1} \text{ for some } q, f_q^{-1}(w), \text{ if } w \in W_{2q} \text{ for some } q.$$

A vertex $T(w)$ is called *dual* for w. If r is odd, then for the first $r-3$ subsets W_i we define dual vertices as described above. For the vertices of the rest 3 subsets, we define two dual vertices, one from every other subset, given by fixed bijections from W_{r-2} to W_{r-1} and from W_{r-1} to W_r.

To prove the existence of an equitable r-coloring of H we shall show the existence a bijection from V to W without monochromatic edges (i.e., for any $A \in E$ and $i = 1, \ldots, r$, $A \nsubseteq W_i$). Let us consider the following independent random elements: a random bijection $g : V \to W$ with uniform distribution; independent random variables $(X_v, v \in V)$ with uniform distribution on $[0, 1]$.

The bijection g induces an r-coloring on V, $v \in V$ is colored with i iff $g(v) \in W_i$ and the duality on V, vertices v and v' are dual iff $g(v)$ and $g(v')$ are dual.

The random variables X_v induce a random ordering σ on V: $\sigma(v) = \sum_{w \in V} I\{X_v \le X_w\}$, where $I\{A\}$ denote the indicator of the event A.

The coloring induced by g can contain monochromatic edges. In this case we shall organize the recoloring process with continuous time $t \in [0, 1]$. Every vertex v is considered only at the time X_v and the following **recoloring condition** is checked:

at the time X_v there is a monochromatic edge A containing v such that v is the first vertex in A in the ordering σ and none of the vertices of A has changed its color before.

If the recoloring condition holds for v, then we change its color to the color of its dual vertex and, vice versa, change the color of the dual vertex to the color of v. Formally, we just replace their images in the bijection g and get another bijection g': if v and v' ar dual, then

$$g'(v) = g(v'), \;\; g'(v') = g(v).$$

So, the colors of dual vertices change if the recoloring condition for at least one of them. In the case of triple dual vertices v, v', v'' we change their colors cyclically: if $g(v) \in W_{r-2}$, $g(v') \in W_{r-1}$, $g(v'') \in W_r$, then

$$g'(v) = g(v'), \;\; g'(v') = g(v''), \;\; g'(v'') = g(v).$$

At the end of the recoloring process we get a random bijection $\widetilde{g} : V \to W$. Our main aim is to show that it induces a proper coloring for the hypergraph H.

Suppose now that the induced by $\widetilde{g}$ the random r-coloring contains a monochromatic edge A of some color α. Due to the construction this edge was not completely colored with α in the initial coloring induced by g. Thus, there should be several vertices in A which were recolored with α during the recoloring stage. Let $v_1, \ldots, v_L$ denote this set of vertices. Our recoloring rule implies that all of them were colored with the fixed color β in the initial coloring.

However, there could be two different types of vertices v_i: the vertex v_i changed its color since the recoloring condition held for v_i; the vertex v_i changed its color since the recoloring condition held for its dual vertex.

So, there are three different cases to consider:

- $L = 1$ and the recoloring condition held for v_1,
- $L = 1$ and the recoloring condition held for the vertex dual to v_1,
- $L > 1$.

Considering all these cases we show that for a small positive constant $c > 0$, the probability of the event that there exist monochromatic edges in the final coloring induced by $\widetilde{g}$ is smaller than 1. So, with positive probability the constructed random coloring will be proper and it will be a required equitable coloring for the hypergraph H.

References

[1] A.M. Raigorodskii, D.A. Shabanov, "The Erdős–Hajnal problem, its generalizations and related problems", *Russian Mathematical Surveys*, **66**:5 (2011), 933–1002

[2] P. Erdős, L. Lovász, "Problems and results on 3-chromatic hypergraphs and some related questions", *Infinite and Finite Sets*, Colloquia Mathematica Societatis Janos Bolyai, North Holland, Amsterdam, **10** (1973), 609–627.

[3] J. Beck, "On 3-chromatic hypergraphs", *Discrete Mathematics*, **24**:2 (1978), 127–137.

[4] D.A. Shabanov, "Coloring non-uniform hypergraphs without short cycles", *Graphs and Combinatorics*, **30**:5 (2014), 1249–1260.

[5] A. Hajnal, E. Szemerédi, "Proof of a conjecture of P. Erdős", *Combinatorial Theory and its Applications* (P. Erdős, A. Hajnal, V.T. Sós), North-Holland, London, 1970, 601–623.

[6] J. Radhakrishnan, A. Srinivasan, "Improved bounds and algorithms for hypergraph two-coloring", *Random Structures and Algorithms*, **16**:1 (2000), 4–32.