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Structural Results for General Partition, Equistable and Triangle graphs

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Abstract

Miklavič and Milanič (2011) introduced the connections among the classes of equistable, general partition and triangle graphs. We present results concerning the three classes aforementioned. In particular, we show that the general partition and triangle classes are both closed under the operations of substitution, induction and contraction of modules. Moreover, we show that the triangle condition is sufficient for a planar graph to be a general partition graph, providing a generalisation of a result by Mahadev, Peled and Sun (1994) on equistable outerplanar graphs.

Keywords: Equistable graph, General partition graph, Triangle graph, Planar graph, Module operations.

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[1] This research was partially supported by CNPq, Brazil.

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1 Introduction

We consider finite simple undirected graphs and follow notations and definitions from Bondy and Murty [2].

The following chain of graph classes inclusions was shown by Miklavič and Milanič [7]:

$$\text{general partition graphs} \subseteq \text{equistable graphs} \subset \text{triangle graphs}.$$

Refuting a conjecture by Orlin (cf. Levit and Milanič [4]), Milanič and Trotignon [8] showed that the class of general partition graphs is properly contained in the class of equistable graphs, strengthening the above chain to:

$$\text{general partition graphs} \subset \text{equistable graphs} \subset \text{triangle graphs}.$$

A graph $G = (V, E)$ is a *general partition graph* if there exists a set U and an assignment of a nonempty subset $U_v \subseteq U$ to each vertex v of G such that two vertices u and v are adjacent if and only if $U_u \cap U_v \neq \emptyset$, and for every maximal independent set S of G, $U = \bigcup_{v \in S} U_v$.

A clique in a graph is a *strong clique* if it has non-empty intersection with each maximal independent set of the graph. Moreover, a collection of cliques is a *clique cover* of G if for every edge of G there is a clique in the collection that contains both its endpoints. Finally, a clique cover is a *strong clique cover* if all of its cliques are strong cliques.

Theorem 1.1 (McAvaney et al. [6]) *A graph G is a general partition graph if and only if there is a strong clique cover of G.*

A graph G satisfies the *triangle condition* if for every maximal independent set S and every edge uv in $G \setminus S$ there is a vertex $s \in S$ such that $\{u, v, s\}$ induces a triangle in G. A graph is a *triangle graph* if it satisfies the triangle condition.

An induced P_4 *abcd* in a graph G is a *bad* P_4 if there exists a maximal independent set S in G containing both a and d such that no vertex in S is simultaneously adjacent to b and c. Such a maximal independent set S is a *witness* for the bad P_4.

Theorem 1.2 (Miklavič and Milanič [7]) *A graph G is a triangle graph if and only if G contains no bad P_4.*

In this work, we show that the general partition and triangle classes of graphs are both closed under the operations of substitution, induction and contraction of modules (the definitions are in the next section), which allows us to get as corollaries results by McAvaney et al. [6] and the lexicographic product result by Miklavič and Milanič [7]. As a further result we show that for planar graphs the triangle condition is sufficient for a graph to be a general partition graph, answering a question rised by Anbeek et al. [1].

2 Module operations

Let G be a graph and $U \subseteq V(G)$. A vertex $x \notin U$ *distinguishes* U if x has both a neighbour and a non-neighbour in U. A subset of vertices of G is a *module* if it is indistinguishable by the vertices that do not belong to it.

Next result warrants that the general partition and triangle classes of graphs are both closed under the operations of induction by a module and contraction of modules.

Theorem 2.1 *Let G be a general partition (triangle) graph, and let M be a module of G. Then, the graph induced by M is a general partition (triangle) graph. Furthermore, the graph obtained from G by replacing M with a single vertex adjacent to the neighbours of M is a general partition (triangle) graph.*

Proof. Consider G a general partition graph, M a module of G and $\mathcal{C}$ a strong clique cover of G. Notice that any strong clique cannot be entirely contained in M. So, we claim that $\mathcal{C}' = \{C \cap M \mid C \in \mathcal{C}\}$ is a clique cover of $G[M]$. Suppose there exists $C' \in \mathcal{C}'$ such that C' is not a strong clique in $G[M]$. This means that there is a maximal independent set S' in $G[M]$ such that $C' \cap S' = \emptyset$. Now, take $C \in \mathcal{C}$ such that $C' \subset C$ and extend S' to a maximal independent set S in G. Because S' has some vertices of M, no vertex adjacent to some vertex of M is in S. So, $C \cap S = \emptyset$, contradicting that C is a strong clique of G. Therefore, $G[M]$ is a general partition graph.

Let G' be the graph obtained by the contraction of M into a single vertex v. It is easy to see that $\mathcal{C}' = \{C \mid C \cap M = \emptyset \text{ and } C \in \mathcal{C}\} \cup \{C \setminus M \cup \{v\} \mid C \cap M \neq \emptyset \text{ and } C \in \mathcal{C}\}$ is a strong clique cover of G', proving that G' is a general partition graph.

Suppose G is a triangle graph and let P be a bad P_4 in $G[M]$ having S_M as a witness. Consider S a maximal independent set of G such that $S_M \subset S$. Since $S \setminus S_M$ cannot have any vertex adjacent to some vertex in M, we have that P is also a bad P_4 in G having S as a witness, a contradiction.

It is not hard to see that the existence of a bad P_4 in the graph obtained by the contraction of M into a single vertex v implies the existence of a bad P_4 in G. □

Let G_1 and G_2 be disjoint graphs and u be a vertex of G_1. Define $G =$

$G_1(u \to G_2)$ as the graph such that $V(G) = V(G_1) \setminus \{u\} \cup V(G_2)$ and $E(G) = E(G_1 - u) \cup E(G_2) \cup E'$ where $v_1v_2 \in E'$ if and only if $v_1 \in V(G_1) \cap N(u)$ and $v_2 \in V(G_2)$.

Next result warrants that the general partition and triangle classes of graphs are both closed under the operation of substitution.

Theorem 2.2 *Let G_1 and G_2 be disjoint graphs, and u be a vertex of G_1. Then G_1 and G_2 are general partition (triangle) graphs if and only if $G_1(u \to G_2)$ is a general partition (triangle) graph.*

Proof. Let $G = G_1(u \to G_2)$ and $G_2 = (V_2, E_2)$. Notice that V_2 is a module of G and G_1 can be obtained from G by replacing V_2 with a single vertex adjacent to the neighbors of V_2. Therefore, by Theorem 2.1, if G is a general partition (triangle) graph, then also are both G_1 and G_2.

Given a graph H, denote respectively by $\mathcal{S}_H$ and $\mathcal{C}_H$, the set of all maximal independent sets and the set of all maximal cliques of H. Notice that by the construction of G, we have $\mathcal{S}_G = \{S_1 \mid u \notin S_1 \text{ and } S_1 \in \mathcal{S}_{G_1}\} \cup \{S_1 \setminus \{u\} \cup S_2 \mid u \in S_1, S_1 \in \mathcal{S}_{G_1} \text{ and } S_2 \in \mathcal{S}_{G_2}\}$. Moreover, $\mathcal{C}_G = \{C_1 \mid u \notin C_1 \text{ and } C_1 \in \mathcal{C}_{G_1}\} \cup \{C_1 \setminus \{u\} \cup C_2 \mid u \in C_1, C_1 \in \mathcal{C}_{G_1}, C_2 \in \mathcal{C}_{G_2}\}$.

Let G_1 and G_2 be general partition graphs with $\mathcal{C}_1$ and $\mathcal{C}_2$ its strong clique covers, respectively. Because of the construction of $\mathcal{S}_G$ and $\mathcal{C}_G$, we have that $\mathcal{C} = \{C_1 \mid u \notin C_1, C_1 \in \mathcal{C}_1\} \cup \{C_1 \setminus \{u\} \cup C_2 \mid u \in C_1,\ C_1 \in \mathcal{C}_1 \text{ and } C_2 \in \mathcal{C}_2\}$ is a strong clique cover of G. Therefore, G is a general partition graph.

Suppose G_1 and G_2 are triangle graphs and consider $G = G_1(u \to G_2)$. Since V_2 is a module of G, the vertices of any induced P_4 in G, with some vertex v in V_2, are either entirely contained in V_2 or v is an end-point of the P_4 and it is the only vertex of the P_4 in V_2. Therefore, any induced P_4 in G can be seen as an induced P_4 in either G_1 or G_2. Now, let P be a bad P_4 in G and let S be a witness for it. If the vertices of P are contained in V_2, then P is a bad P_4 in G_2 having $S_2 = S \cap V_2$ as its witness. If the intersection of the vertices of P with V_2 contains only v, then switch v for u in P and $S \setminus V_2 \cup \{u\}$ is a witness for the new bad P_4 in G_1. At last, if the vertices of P are contained in V_1, then either S or $S \setminus V_2 \cup \{u\}$ is a witness for P being a bad P_4 in G_1. $\square$

3 Planar graphs

A graph G is planar if it has a planar representation with no crossing edges. In this section we show that the triangle condition is sufficient to a planar graph be a general partition graph.

Theorem 3.1 (Kuratowski [2]) *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 3.2 *Let G be a planar graph satisfying the triangle condition. Then, G is a general partition graph.*

Proof. Suppose G is not a general partition graph. Then, there exists an edge $xy \in E(G)$ such that every maximal clique containing $\{x, y\}$ is not a strong clique.

Let $t = |N(x) \cap N(y)|$. Consider a planar representation of G in which xy is in the outer face and name the vertices in $N(x) \cap N(y)$ by $v_1, \ldots, v_t$, in a manner that the vertices in $\{v_1, \ldots, v_{j-1}\}$ are inside the region defined by the cycle xyv_j, $1 < j \leq t$.

Notice that in this planar representation of G, the neighbourhood of v_j, $1 < j < t$, can only be found in the interior of the closed regions defined by the cycles $xv_{j-1}yv_j$ and xv_jyv_{j+1}.

Let $C_1, C_2, \ldots, C_k$ be the maximal cliques containing $\{x, y\}$. Due to the labeling of the shared neighbourhood of x and y, if $|C_i| = 3$, then $C_i = \{x, y, v_j\}$ for some j in $\{1, \ldots, t\}$. And if $|C_i| = 4$, then $C_i = \{x, y, v_j, v_{j+1}\}$, $1 \leq j < t$.

Since C_i is not a strong clique, there is a maximal independent S such that $C_i \cap S = \emptyset$. Moreover, S has vertices a_i and b_i such that a_i is adjacent to both x and v_j, and b_i is adjacent to both v_j and y. Notice that if C_i has size 3, then a_i and b_i are necessarily different. Furthermore, when C_i has size 4, it follows that S will also have a vertex m_i adjacent to both v_j and v_{j+1}. Besides, if there exists a vertex adjacent to both v_j and a_i that is different from x and does not belong to $N(x) \cap N(y)$, denote it by v_{a_i}. Analogously, denote by v_{b_i} a vertex adjacent to both v_j and b_i, different from y and all v_j's.

Proposition 3.3 *There exists an independent set I in G satisfying the following properties:*

(i) *Every $u \in I$, is a a_i, b_i, m_i, v_{a_i} or v_{b_i} for some i in $\{1, \ldots, k\}$;*

(ii) *Each $v_j \in N(x) \cap N(y)$ has a neighbour in I;*

(iii) *For some $1 \leq r, s \leq k$, the vertices $a_r, b_s \in I$;*

(iv) *$I \cap N[x] \cap N[y] = \emptyset$.*

We omit the proof of this proposition.

Therefore, by extending such I to a maximal independent set I' of G, we obtain a maximal independent set of G that is disjoint with all the cliques containing both x and y. That means no vertex in I' is adjacent to both end points of the edge xy, contradicting that G satisfies the triangular condition.□

Corollary 3.4 *For a planar graph G, the following are equivalent:*

(i) *G is a triangle graph;*

(ii) *G is an equistable graph;*

(iii) *G is a general partition graph.*

Since Kloks et al. [3] presented a polynomial-time algorithm that verifies if a planar graph satisfies the triangle condition, we have that triangle/equistable/general partition graphs can be recognized in polynomial time.

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