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First non-trivial upper bound on circular chromatic number of the plane.

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Abstract

In this paper we consider a circular version of the Nelson-Hadwiger problem. Namely, we show a circular $(4+\frac{4\sqrt{3}}{3})$-coloring of the unit distance graph, which is a graph with the set of all points of the plane as the vertex set and any two points adjacent if their euclidean distance is equal to one.

Keywords: Nelson-Hadwiger problem, circular coloring, coloring of the plane

We refer to the famous Nelson-Hadwiger problem and a well studied coloring model, which is circular coloring. The Nelson-Hadwiger problem is the question for the chromatic number of the plane, which is the minimum number of colors required to color every point of the plane in such a way that no two points at distance 1 from each other have the same color. The exact answer to the question is not known. We only know that at least 4 colors are needed [5] and 7 colors suffice [3]. For a comprehensive history of the Hadwiger-Nelson problem see the monograph by Soifer [7].

An r-circular coloring of a graph $G=(V,E)$ is a function $c\colon V\to[0,r)$ such that for any edge uv of G holds $1\le|c(u)-c(v)|\le r-1$. Notice that

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an r-circular coloring can be seen as an assignment of arcs of length 1 of the circle with perimeter r to vertices of G in such a way that adjacent vertices get disjoint arcs. The circular chromatic number of a graph G is the number $\chi_c(G) = \inf\{r \in \mathbb{R} : \text{ there exists } r\text{-circular coloring of } G\}$. Circular coloring was first introduced by Vince [8]. For a survey see Zhu [9]. It is known [8] that the circular chromatic number does not exceed the chromatic number, but is bigger than the chromatic number minus one ($\chi_c(G) \in (\chi(G)-1, \chi(G)]$ for any graph G).

By r-circular coloring of the plane we mean a function $c : \mathbb{R}^2 \to [0, r)$ such that for any two points u, v which are at distance 1 holds: $1 \leq |c(u) - (v)| \leq r - 1$. By the circular chromatic number of the plane we mean $\chi_c(\mathbb{R}^2) = \inf\{r \in \mathbb{R} : \text{ there exists } r\text{-circular coloring of the plane}\}$. If we combine known bounds for the chromatic number of the plane with properties of the circular chromatic number, we obtain $3 < \chi_c(\mathbb{R}^2) \leq 7$. The lower bound can be improved by comparing with another interesting parameter, i.e. the fractional chromatic number. To define it we first need to define the so-called j-fold coloring of the plane. A function is a j-fold coloring if it assigns a j-element set of natural numbers to every point of the plane in such a way that points in distance 1 get disjoint sets. The fractional chromatic number of the plane $\chi_f(\mathbb{R}^2)$ is defined by

$$\chi_f(\mathbb{R}^2) = \inf\{\frac{k}{j} : \text{ there exists } j-\text{fold coloring of the plane using } k \text{ colors.}\}$$

It is known [9] that $\chi_f(\mathbb{R}^2) \leq \chi_c(\mathbb{R}^2)$. The best known lower bound for $\chi_f(\mathbb{R}^2)$ is $\frac{32}{9} = 3.555..$ and it can be found in the book by Scheinerman and Ullman [6]. This gives us a lower bound on the circular chromatic number of the plane. DeVos *et al.* [1] improved this bound by showing that the chromatic number of the plane is at least 4. In this paper we give the first non-trivial upper bound on the circular chromatic number of the plane:

Theorem 0.1

$$\chi_c(\mathbb{R}^2) \leq 4 + \frac{4\sqrt{3}}{3} \leq 6.3095$$

To present a $(4+\frac{4\sqrt{3}}{3})$-circular coloring of the plane we need few definitions. For $x \in \mathbb{R}$ and $\ell \in \mathbb{R}_+$ we define $\lfloor x \rfloor_\ell = \lfloor \frac{x}{\ell} \rfloor \cdot \ell$ and $(x)_\ell = x - \lfloor x \rfloor_\ell$. Notice that for $\ell = 1$, the function $\lfloor x \rfloor_\ell$ is the standard floor function $\lfloor x \rfloor$. Let $\ell = 2+2\sqrt{3}$ and let $r = \frac{2\sqrt{3}}{3}\ell = 4 + \frac{4\sqrt{3}}{3}$.

Let us start with some intuition on the construction of an r-circular coloring of the plane. Let R denote the rectangle $[0,\ell) \times [0,\frac{1}{2})$. We define an r-circular coloring of the rectangle R by $c(x,y) = \frac{2\sqrt{3}}{3}x$, (where $(x,y) \in R$). Then we extend this coloring in a circular way on a strip $S = \mathbb{R} \times [0,\frac{1}{2})$. We simply join copies of the rectangle R by their vertical sides so they form the strip S. Each copy of R is colored in the same way as the original rectangle R. Then we take copies of the strip S and join them with horizontal sides. Each strip S is colored in the same way as the original one, but we shift each copy of S by $(1+\frac{\sqrt{3}}{2})$ to the right, comparing to the strip below (see Figure 1)

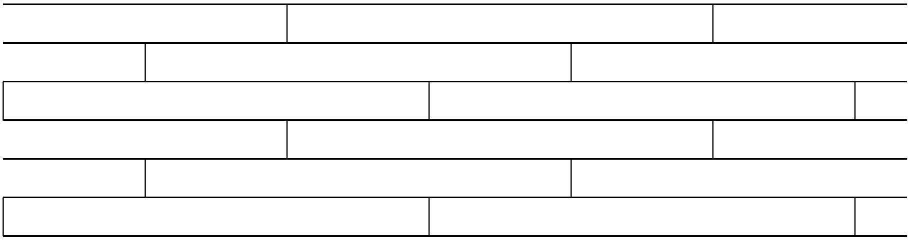

Fig. 1. Partitions of the plane into copies of the rectangle R

Formally, the r-circular colouring of the plane is defined by:

$$c(x,y) = \frac{2\sqrt{3}}{3}\left(x - (2+\sqrt{3})\lfloor y \rfloor_{\frac{1}{2}}\right)_{\ell}.$$

For visualization of the coloring see the Figure 2.

Exoo considered more restricted coloring of the plane in [2]. He asked for the minimum number of colors needed to color the plane in such a way that any two points in at distance belonging to a given interval $[1-\epsilon, 1+\epsilon]$ get different colors. For $\epsilon = 0$ the problem reduces to the Nelson-Hadwiger problem. The fractional and j-fold Exoo-type coloring was studied in [4]. The method of circular coloring of the plane presented in this paper can be adapted to Exoo-type coloring of the plane.

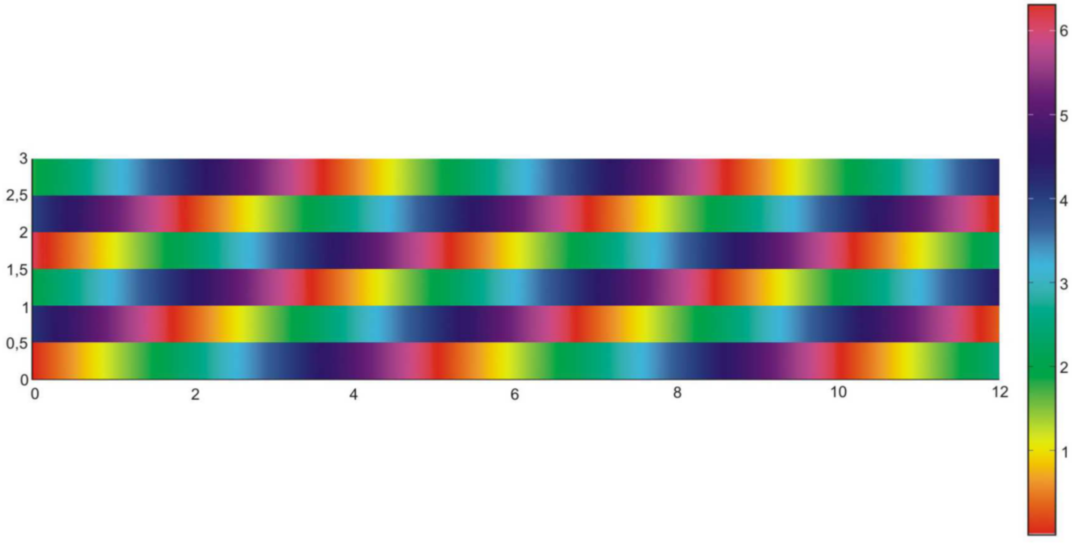

Fig. 2. The $(4+\frac{4\sqrt{3}}{3})$-circular colouring of the plane

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