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Large unavoidable subtournaments

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Abstract

Let D_k denote the tournament on $3k$ vertices consisting of three disjoint vertex classes V_1, V_2 and V_3 of size k, each of which is oriented as a transitive subtournament, and with edges directed from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. Fox and Sudakov proved that given a natural number k and $\epsilon > 0$ there is $n_0(k,\epsilon)$ such that every tournament of order $n \geq n_0(k,\epsilon)$ which is ϵ-far from being transitive contains D_k as a subtournament. Their proof showed that $n_0(k,\epsilon) \leq \epsilon^{-O(k/\epsilon^2)}$ and they conjectured that this could be reduced to $n_0(k,\epsilon) \leq \epsilon^{-O(k)}$. Here we outline a proof of this conjecture.

Keywords: Tournament, Ramsey Theory, Extremal Graph Theory.

1 Introduction

A central result in the Ramsey theory is Ramsey's theorem [10], which says that given any natural number k, there is an integer N such that every two colouring of the edges of the complete graph K_N contains a monochromatic copy of K_k. An important problem in the area is to estimate the smallest

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value of N for which the theorem holds, denoted $R(k)$. It is known that $2^{(1/2+o(1))k} \leq R(k) \leq 4^{(1+o(1))k}$ (see [3], [11], [5], [1]).

For general two colourings of K_N one clearly cannot guarantee any coloured subgraph other than a monochromatic clique in Ramsey's theorem. Bollobás raised the question of which coloured subgraphs occur in two colourings of K_N where each colour appears on at least ϵ proportion of the edges. Let $\mathcal{F}_k$ denote the collection of two coloured graphs of order $2k$, in which one colour appears as either a clique of order k or two disjoint cliques of order k. Bollobás asked whether, given a natural k and $\epsilon > 0$ there is $M = M(k, \epsilon)$ with the following property: in every two colouring of the edges of K_M containing both colours on at least ϵ proportion of the edges, some element of $\mathcal{F}_k$ appears as a coloured subgraph. Cutler and Montágh [2] answered this question in the affirmative and proved that it is possible to take $M(k, \epsilon) \leq 4^{k/\epsilon}$. Fox and Sudakov [6] subsequently improved this bound to show that $M(k, \epsilon) \leq \epsilon^{-ck}$, for some constant $c > 0$. As shown in [6], this bound is tight up to the value of the constant c in the exponent, which can be seen by taking a random two colouring of a graph on $\epsilon^{-(k-1)/2}$ vertices with appropriate densities.

Here we will be concerned with an analogous question for tournaments. A *tournament* is a directed graph obtained by assigning a direction to the edges of a complete graph. A tournament is said to be *transitive* if it is possible to order the vertices of the tournament so that all of its edges point in the same direction. Let $T(k)$ denote the smallest integer such that every tournament on $T(k)$ vertices contains a transitive subtournament on k vertices. A classic result due to Erdős and Moser [4] shows that $T(k)$ is finite for all k and gives that $2^{(k-1)/2} \leq T(k) \leq 2^{k-1}$.

As in the two colouring graph case, it is natural ask which subtournaments must occur in large tournament which is 'not too similar' to a transitive tournament. An n-vertex tournament T is *ϵ-far from being transitive* if in any ordering of the vertices of T, the direction of at least ϵn^2 edges of T must be switched in order to obtain a transitive tournament. In [6], Fox and Sudakov asked the following question: given $\epsilon > 0$, which subtournaments must an n-vertex tournament which is ϵ-far from being transitive contain?

For any natural number k, let D_k denote the tournament on $3k$ vertices consisting of three disjoint vertex classes V_1, V_2 and V_3 of size k, each of which is oriented as a transitive subtournament, and with all edges directed from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. Taking $T = D_{n/3}$ we obtain an n-vertex tournament which is $\frac{1}{9}$-far from being transitive and whose only subtournaments are contained in D_k for some k. Thus, subtournaments of D_k are the only candidates for unavoidable tournaments which occur in large tournaments that are ϵ-far from transitive for small ϵ.

Theorem 1.1 (Fox–Sudakov) *Given $\epsilon > 0$ and a natural number k, there is $n_0(k, \epsilon)$ such that if T is a tournament on $n \geq n_0(k, \epsilon)$ vertices which is ϵ-far from being transitive, then T contains D_k as a subtournament. Furthermore $n_0(k, \epsilon) \leq \epsilon^{-ck/\epsilon^2}$, for some absolute constant $c > 0$.*

The authors in [6] conjectured that this bound can be further reduced to $n_0(k, \epsilon) \leq \epsilon^{-Ck}$ for some absolute constant $C > 0$. This order of growth agrees with high probability with a random tournament obtained by directing edges backwards independently with probability $\approx \epsilon$. Here we prove this conjecture.

Theorem 1.2 *There is a constant $C > 0$ such that for $\epsilon > 0$ and any natural number k we have $n_0(k, \epsilon) \leq \epsilon^{-Ck}$.*

Notation: Given a tournament T, we write $V(T)$ to denote its vertex set and $E(T)$ to denote the directed edge set of T. Given $v \in V(T)$ and a set $S \subset V(T)$, let $d_S^-(v) := |\{u \in S : \overrightarrow{uv} \in E(T)\}|$ and $d_S^+(v) := |\{u \in S : \overrightarrow{vu} \in E(T)\}|$. We will also write $T[S]$ to denote the induced subtournament of T on vertex set S. Given $B \subset E(T)$, we write $d_B^-(v) = |\{u \in V(T) : \overrightarrow{uv} \in B\}|$ and $d_B^+(v) = |\{u \in V(T) : \overrightarrow{uv} \in B\}|$. For an ordering $v_1, \ldots, v_{|T|}$ of $V(T)$ and $1 \leq i < j \leq |T|$, let $[v_i, v_j] := \{v_i, v_{i+1}, \ldots, v_j\}$. Lastly, all log functions will be to the base 2.

2 Outline of the proof of Theorem 1.2

2.1 Finding many long backwards edges in T

In [6], Theorem 1.1 was deduced from two results of independent interest. The first result showed that any tournament which is ϵ-far from being transitive must contain many directed triangles.

Theorem 2.1 (Theorem 1.3 [6]) *Any n-vertex tournament T which is ϵ-far from being transitive contains at least $c\epsilon^2 n^3$ directed triangles, where $c > 0$ is an absolute constant.*

As pointed out in [6], this bound is also best possible in general, as can be seen from the following tournament. Let T be given by taking k copies of $D_{n/3k}$, say on disjoint vertex sets $V_1, \ldots, V_k$ with all edges between V_i and V_j directed forward, for $i < j$. As at least $(n/3k)^2$ edges from each copy of $D_{n/3k}$ must be reoriented in order to obtain a transitive tournament, T is $k(1/3k)^2 = 1/9k$ far from being transitive, but contains only $k.(n/3k)^3 = n^3/27k^2$ directed triangles. Taking $\epsilon = 1/9k$, we see that the growth rate here agrees with that given by Theorem 2.1 up to constants.

Our first improvement in the bound for $n_0(k,\epsilon)$ comes from showing that any tournament which is ϵ-far from being transitive must either contain many more directed triangles than the number given in Theorem 2.1 or contain a slightly smaller subtournament which is 2ϵ-far from being transitive.

Given an ordering $v_1, \ldots, v_{|T|}$ of the vertices of a tournament T, edges of the form $\overleftarrow{v_i v_j}$ with $i < j$ are called *backwards edges*. We will often list the vertices of tournaments in an order which minimizes the number of backwards edges. Such orderings are said to be *optimal*.

Proposition 2.2 *Suppose that T is a tournament on n vertices and $v_1, \ldots, v_n$ is an optimal ordering of $V(T)$. Then the following hold:*

(i) *For every $i, j \in [n]$ with $i < j$ we have*
 - $d^+_{[v_{i+1},v_j]}(v_i) \geq (j-i)/2$;
 - $d^-_{[v_i,v_{j-1}]}(v_j) \geq (j-i)/2$.

(ii) *If $T[v_{i+1}, v_j] := T[\{v_{i+1}, \ldots, v_j\}]$ has $\delta(j-i)^2$ backwards edges in this ordering then the subtournament $T[v_{i+1}, v_j]$ is δ-far from being transitive.*

Given an ordering $v_1, \ldots, v_n$ of $V(T)$ with a backwards edge $\overleftarrow{v_i v_j}$ $(i < j)$, the edge $\overleftarrow{v_i v_j} \in B$ is said to have *length* $j - i$.

Lemma 2.3 *Suppose that T is a tournament on n vertices which is ϵ-far from being transitive and let $v_1, \ldots, v_n$ be an optimal ordering of $V(T)$. Let B denote the collection of backwards edges in this ordering. Then one of the following holds:*

(i) *The subset B' of B consisting of those edges of length at least $n/16$ satisfies $|B'| \geq |B|/4$;*

(ii) *T contains a subtournament on at least $n/8$ vertices which is 2ϵ-far from being transitive.*

The next lemma shows that if we are in case 2. of Lemma 2.3 and ϵ isn't too large then there is a large set of backwards edges of T all of which lie in a huge number of directed triangles.

Lemma 2.4 *Let T be an n-vertex tournament with optimal ordering $v_1, \ldots, v_n$ and let B denote the set of backwards edges in this ordering, $|B| = \alpha n^2$. Suppose that the subset $B' \subset B$ of backwards edges with length at least $n/16$ satisfies $|B'| \geq \alpha n^2/4$. Then, provided that $\alpha \leq 2^{-16}$, there exists $B'' \subset B'$ satisfying $|B''| \geq |B'|/2$ with the property that each edge of B'' lies in at least*

$n/64$ *directed triangles in* T.

Proof. Sketch Note that given $\overleftarrow{x_i x_j} \in B'$ with $i < j$, if k satisfies $i < k < j$ and both $\overrightarrow{x_i x_k}$ and $\overrightarrow{x_k x_j}$ are edges of T, the set $\{x_i, x_k, x_j\}$ forms a directed triangle in T. Note also that by Proposition 2.2 there are at least $(j-i)/2$ edges $\overrightarrow{x_i x_k}$ in T with $i < k < j$. In order to block such a 'potential directed triangle', the edge $\overleftarrow{x_k x_j}$ must lie in T. Thus there must be many edges in B which are directed away from x_j. An identical argument shows that there must be many backwards edges directed towards x_i. But as each edge in B' has many 'potential threats' – $(j-i)/2$ of them. By using that B is small, it can be shown that most of these potential directed triangles are not blocked and form directed triangles, as required. □

2.2 Finding a copy of D_k in T

The second half of our argument is based on another result from [6]. Here the authors proved that the following holds:

Theorem 2.5 (Theorem 3.5, [6]) *Any* n*-vertex tournament with at least* δn^3 *directed triangle contains* D_k *as a subtournament provided that* $n \geq \delta^{-4k/\delta}$.

By combining Lemma 2.3 and Lemma 2.4 with Theorem 2.5 it is already possible to improve the bound $n_0(k, \epsilon)$, to show that $n_0(k, \epsilon) \leq \epsilon^{-ck/\epsilon}$ for some fixed constant $c > 0$. To remove the additional ϵ term from the exponent, we modify Theorem 2.5.

The next lemma shows that if many directed triangles in Theorem 2.5 occur in a very unbalanced manner, meaning that each of these triangles contain an edge from a small set, the lower bound on n in Theorem 2.5 can be reduced. Note that this is exactly the situation given by Lemma 2.4.

Lemma 2.6 *Let* T *be an* n*-vertex tournament and let* E *be a set of edges of* βn^2 *edges in* T. *Suppose that each edge of* E *occurs in at least* γn *directed triangles in* T. *Then* T *contains* D_k *as a subtournament provided* $n \geq \beta^{-100k/\gamma}$.

The proof of Lemma 2.6 follows a similar outline to that of Theorem 2.5 in [6].

Theorem 1.2 can now be proven as follows. Let T be an n-vertex tournament which is ϵ-far from being transitive. By repeatedly applying Lemma 2.3 we can find a sequence of subtournaments $T_1, \ldots, T_L$, with $|T_i| \geq n/8^i$ for all i such that T_i is $2^i\epsilon$ -far from being transitive. When this process terminates at T_L, by Lemma 2.3 part 2. T_L must contain many 'long backwards edges'. We then apply Lemma 2.6 to find a large set of backwards edges in T_L, each of which occurs in many directed triangles. The theorem is then proven by applying Theorem 2.6 to this set.

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