Large unavoidable subtournaments

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Abstract
Let $D_k$ denote the tournament on $3k$ vertices consisting of three disjoint vertex classes $V_1$, $V_2$ and $V_3$ of size $k$, each of which is oriented as a transitive subtournament, and with edges directed from $V_1$ to $V_2$, from $V_2$ to $V_3$ and from $V_3$ to $V_1$. Fox and Sudakov proved that given a natural number $k$ and $\epsilon > 0$ there is $n_0(k, \epsilon)$ such that every tournament of order $n \geq n_0(k, \epsilon)$ which is $\epsilon$-far from being transitive contains $D_k$ as a subtournament. Their proof showed that $n_0(k, \epsilon) \leq \epsilon^{-O(k/\epsilon^2)}$ and they conjectured that this could be reduced to $n_0(k, \epsilon) \leq \epsilon^{-O(k)}$. Here we outline a proof of this conjecture.

Keywords: Tournament, Ramsey Theory, Extremal Graph Theory.

1 Introduction

A central result in the Ramsey theory is Ramsey’s theorem [10], which says that given any natural number $k$, there is an integer $N$ such that every two colouring of the edges of the complete graph $K_N$ contains a monochromatic copy of $K_k$. An important problem in the area is to estimate the smallest
value of $N$ for which the theorem holds, denoted $R(k)$. It is known that $2^{(1/2+o(1))k} \leq R(k) \leq 4^{(1+o(1))k}$ (see [3], [11], [5], [1]).

For general two colourings of $K_N$ one clearly cannot guarantee any coloured subgraph other than a monochromatic clique in Ramsey’s theorem. Bollobás raised the question of which coloured subgraphs occur in two colourings of $K_N$ where each colour appears on at least $\epsilon$ proportion of the edges. Let $\mathcal{F}_k$ denote the collection of two coloured graphs of order $2k$, in which one colour appears as either a clique of order $k$ or two disjoint cliques of order $k$. Bollobás asked whether, given a natural $k$ and $\epsilon > 0$ there is $M = M(k, \epsilon)$ with the following property: in every two colouring of the edges of $K_M$ containing both colours on at least $\epsilon$ proportion of the edges, some element of $\mathcal{F}_k$ appears as a coloured subgraph. Cutler and Montágh [2] answered this question in the affirmative and proved that it is possible to take $M(k, \epsilon) \leq 4^{k/\epsilon}$. Fox and Sudakov [6] subsequently improved this bound to show that $M(k, \epsilon) \leq e^{-ck}$, for some constant $c > 0$. As shown in [6], this bound is tight up to the value of the constant $c$ in the exponent, which can be seen by taking a random two colouring of a graph on $\epsilon^{-k-1/2}$ vertices with appropriate densities.

Here we will be concerned with an analogous question for tournaments. A tournament is a directed graph obtained by assigning a direction to the edges of a complete graph. A tournament is said to be transitive if it is possible to order the vertices of the tournament so that all of its edges point in the same direction. Let $T(k)$ denote the smallest integer such that every tournament on $T(k)$ vertices contains a transitive subtournament on $k$ vertices. A classic result due to Erdős and Moser [4] shows that $2^{(k-1)/2} \leq T(k) \leq 2^k - 1$.

As in the two colouring graph case, it is natural ask which subtournaments must occur in large tournament which is ‘not too similar’ to a transitive tournament. An $n$-vertex tournament $T$ is $\epsilon$-far from being transitive if in any ordering of the vertices of $T$, the direction of at least $en^2$ edges of $T$ must be switched in order to obtain a transitive tournament. In [6], Fox and Sudakov asked the following question: given $\epsilon > 0$, which subtournaments must an $n$-vertex tournament which is $\epsilon$-far from being transitive contain?

For any natural number $k$, let $D_k$ denote the tournament on $3k$ vertices consisting of three disjoint vertex classes $V_1$, $V_2$ and $V_3$ of size $k$, each of which is oriented as a transitive subtournament, and with all edges directed from $V_1$ to $V_2$, from $V_2$ to $V_3$ and from $V_3$ to $V_1$. Taking $T = D_{n/3}$ we obtain an $n$-vertex tournament which is $\frac{1}{9}$-far from being transitive and whose only subtournaments are contained in $D_k$ for some $k$. Thus, subtournaments of $D_k$ are the only candidates for unavoidable tournaments which occur in large
tournaments that are $\epsilon$-far from transitive for small $\epsilon$.

**Theorem 1.1 (Fox–Sudakov)** Given $\epsilon > 0$ and a natural number $k$, there is $n_0(k, \epsilon)$ such that if $T$ is a tournament on $n \geq n_0(k, \epsilon)$ vertices which is $\epsilon$-far from being transitive, then $T$ contains $D_k$ as a subtournament. Furthermore $n_0(k, \epsilon) \leq \epsilon^{-ck/\epsilon^2}$, for some absolute constant $c > 0$.

The authors in [6] conjectured that this bound can be further reduced to $n_0(k, \epsilon) \leq \epsilon^{-Ck}$ for some absolute constant $C > 0$. This order of growth agrees with high probability with a random tournament obtained by directing edges backwards independently with probability $\approx \epsilon$. Here we prove this conjecture.

**Theorem 1.2** There is a constant $C > 0$ such that for $\epsilon > 0$ and any natural number $k$ we have $n_0(k, \epsilon) \leq \epsilon^{-Ck}$.

Notation: Given a tournament $T$, we write $V(T)$ to denote its vertex set and $E(T)$ to denote the directed edge set of $T$. Given $v \in V(T)$ and a set $S \subset V(T)$, let $d_S^-(v) := |\{u \in S : \overrightarrow{vu} \in E(T)\}|$ and $d_S^+(v) := |\{u \in S : \overrightarrow{uv} \in E(T)\}|$. We will also write $T[S]$ to denote the induced subtournament of $T$ on vertex set $S$. Given $B \subset E(T)$, we write $d_B(v) = |\{u \in V(T) : \overrightarrow{vu} \in B\}|$ and $d_B^-(v) = |\{u \in V(T) : \overrightarrow{uv} \in B\}|$. For an ordering $v_1, \ldots, v_{|T|}$ of $V(T)$ and $1 \leq i < j \leq |T|$, let $[v_i, v_j] := \{v_i, v_{i+1}, \ldots, v_j\}$. Lastly, all log functions will be to the base 2.

2 Outline of the proof of Theorem 1.2

2.1 Finding many long backwards edges in $T$

In [6], Theorem 1.1 was deduced from two results of independent interest. The first result showed that any tournament which is $\epsilon$-far from being transitive must contain many directed triangles.

**Theorem 2.1 (Theorem 1.3 [6])** Any $n$-vertex tournament $T$ which is $\epsilon$-far from being transitive contains at least $c\epsilon^2 n^3$ directed triangles, where $c > 0$ is an absolute constant.

As pointed out in [6], this bound is also best possible in general, as can be seen from the following tournament. Let $T$ be given by taking $k$ copies of $D_{n/3k}$, say on disjoint vertex sets $V_1, \ldots, V_k$ with all edges between $V_i$ and $V_j$ directed forward, for $i < j$. As at least $(n/3k)^2$ edges from each copy of $D_{n/3k}$ must be reoriented in order to obtain a transitive tournament, $T$ is $k(1/3k)^2 = 1/9k$ far from being transitive, but contains only $k.(n/3k)^3 = n^3/27k^2$ directed...
triangles. Taking $\epsilon = 1/9k$, we see that the growth rate here agrees with that given by Theorem 2.1 up to constants.

Our first improvement in the bound for $n_0(k, \epsilon)$ comes from showing that any tournament which is $\epsilon$-far from being transitive must either contain many more directed triangles than the number given in Theorem 2.1 or contain a slightly smaller subtournament which is $2\epsilon$-far from being transitive.

Given an ordering $v_1, \ldots, v_{|T|}$ of the vertices of a tournament $T$, edges of the form $\overrightarrow{v_i v_j}$ with $i < j$ are called backwards edges. We will often list the vertices of tournaments in an order which minimizes the number of backwards edges. Such orderings are said to be optimal.

**Proposition 2.2** Suppose that $T$ is a tournament on $n$ vertices and $v_1, \ldots, v_n$ is an optimal ordering of $V(T)$. Then the following hold:

(i) For every $i, j \in [n]$ with $i < j$ we have
   - $d^+_{v_{i+1}, v_j}(v_i) \geq (j - i)/2$;
   - $d^-_{v_{i}, v_{j-1}}(v_j) \geq (j - i)/2$.

(ii) If $T[v_{i+1}, v_j] := T[\{v_{i+1}, \ldots, v_j\}]$ has $\delta(j - i)^2$ backwards edges in this ordering then the subtournament $T[v_{i+1}, v_j]$ is $\delta$-far from being transitive.

Given an ordering $v_1, \ldots, v_n$ of $V(T)$ with a backwards edge $\overrightarrow{v_i v_j}$ ($i < j$), the edge $\overrightarrow{v_i v_j} \in B$ is said to have length $j - i$.

**Lemma 2.3** Suppose that $T$ is a tournament on $n$ vertices which is $\epsilon$-far from being transitive and let $v_1, \ldots, v_n$ be an optimal ordering of $V(T)$. Let $B$ denote the collection of backwards edges in this ordering. Then one of the following holds:

(i) The subset $B'$ of $B$ consisting of those edges of length at least $n/16$ satisfies $|B'| \geq |B|/4$;

(ii) $T$ contains a subtournament on at least $n/8$ vertices which is $2\epsilon$-far from being transitive.

The next lemma shows that if we are in case 2. of Lemma 2.3 and $\epsilon$ isn’t too large then there is a large set of backwards edges of $T$ all of which lie in a huge number of directed triangles.

**Lemma 2.4** Let $T$ be an $n$-vertex tournament with optimal ordering $v_1, \ldots, v_n$ and let $B$ denote the set of backwards edges in this ordering, $|B| = \alpha n^2$. Suppose that the subset $B' \subset B$ of backwards edges with length at least $n/16$ satisfies $|B'| \geq \alpha n^2/4$. Then, provided that $\alpha \leq 2^{-16}$, there exists $B'' \subset B'$ satisfying $|B''| \geq |B'|/2$ with the property that each edge of $B''$ lies in at least
directed triangles in $T$.

**Proof.** Sketch Note that given $\overrightarrow{x_i x_j} \in B'$ with $i < j$, if $k$ satisfies $i < k < j$ and both $\overrightarrow{x_i x_k}$ and $\overrightarrow{x_k x_j}$ are edges of $T$, the set $\{x_i, x_k, x_j\}$ forms a directed triangle in $T$. Note also that by Proposition 2.2 there are at least $(j - i)/2$ edges $\overrightarrow{x_i x_k}$ in $T$ with $i < k < j$. In order to block such a ‘potential directed triangle’, the edge $\overrightarrow{x_k x_j}$ must lie in $T$. Thus there must be many edges in $B$ which are directed away from $x_j$. An identical argument shows that there must be many backwards edges directed towards $x_i$. But as each edge in $B'$ has many ‘potential threats’ – $(j - i)/2$ of them. By using that $B$ is small, it can be shown that most of these potential directed triangles are not blocked and form directed triangles, as required. \(\square\)

### 2.2 Finding a copy of $D_k$ in $T$

The second half of our argument is based on another result from [6]. Here the authors proved that the following holds:

**Theorem 2.5** *(Theorem 3.5, [6])* Any $n$-vertex tournament with at least $\delta n^3$ directed triangle contains $D_k$ as a subtournament provided that $n \geq \delta^{-4k/\delta}$.

By combining Lemma 2.3 and Lemma 2.4 with Theorem 2.5 it is already possible to improve the bound $n_0(k, \epsilon)$, to show that $n_0(k, \epsilon) \leq \epsilon^{-ck/\epsilon}$ for some fixed constant $c > 0$. To remove the additional $\epsilon$ term from the exponent, we modify Theorem 2.5.

The next lemma shows that if many directed triangles in Theorem 2.5 occur in a very unbalanced manner, meaning that each of these triangles contain an edge from a small set, the lower bound on $n$ in Theorem 2.5 can be reduced. Note that this is exactly the situation given by Lemma 2.4.

**Lemma 2.6** Let $T$ be an $n$-vertex tournament and let $E$ be a set of edges of $\beta n^2$ edges in $T$. Suppose that each edge of $E$ occurs in at least $\gamma n$ directed triangles in $T$. Then $T$ contains $D_k$ as a subtournament provided $n \geq \beta^{-100k/\gamma}$.

The proof of Lemma 2.6 follows a similar outline to that of Theorem 2.5 in [6].

Theorem 1.2 can now be proven as follows. Let $T$ be an $n$-vertex tournament which is $\epsilon$-far from being transitive. By repeatedly applying Lemma 2.3 we can find a sequence of subtournaments $T_1, \ldots, T_L$, with $|T_i| \geq n/8^i$ for all $i$ such that $T_i$ is $2^i \epsilon$ -far from being transitive. When this process terminates at $T_L$, by Lemma 2.3 part 2. $T_L$ must contain many ‘long backwards edges’. We then apply Lemma 2.6 to find a large set of backwards edges in $T_L$, each
of which occurs in many directed triangles. The theorem is then proven by applying Theorem 2.6 to this set.

References


