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Ramsey classes with forbidden homomorphisms and a closure

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1 Introduction

Extending classical early Ramsey-type results, the structural Ramsey theory originated at the beginning of 70ies, see [11] for references. However the list of Ramsey classes, as the top of the line of Ramsey properties, was somewhat limited. This was also supported by result of Nešetřil [7] that made a connection between Ramsey classes and ultrahomogeneous structures showing particularly that there are only four types of Ramsey classes of undirected graphs. This connection led to the classification programme for Ramsey classes [8] and, perhaps more importantly, to the connection to the topological dynamics and ergodic theory [6].

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Let us start with the key definition of this paper. Let $\mathcal{K}$ be a class of structures endowed with embeddings. For objects $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all sub-objects $\widetilde{\mathbf{A}}$ of $\mathbf{B}$, $\widetilde{\mathbf{A}}$ isomorphic to $\mathbf{A}$. (By a sub-object we mean that the inclusion is an embedding.) Using this notation the definition of Ramsey class gets the following form: A class $\mathcal{K}$ is a *Ramsey class* if for every its two objects $\mathbf{A}$ and $\mathbf{B}$ and for every positive integer k there exists object $\mathbf{C} \in \mathcal{K}$ such that the following holds: For every partition $\binom{\mathbf{B}}{\mathbf{A}}$ in k classes there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition. It is usual to shorten the last part of the definition as $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

It is not known which classes $\mathcal{K}$ are Ramsey but there are often trivial obstacles for this. One of them is the lack of rigidity of $\mathcal{K}$. One can expand the structure by new relations (such as ordering of elements) and to use this to definite partition of sub-objects. This trick leads to study of enriched classes $\mathcal{K}'$ induced on $\mathcal{K}$. Thus, for example, instead of dealing with graphs $\mathcal{G}$ we deal with the class of all ordered graphs $\overrightarrow{\mathcal{G}}$.

Structures with this additional informations are called lifts and the basic question asked during the Bertinoro 2011 meeting by several people is whether every ultrahomogeneous structure (and equivalently every ω-categorical structure) has a finitary Ramsey lift. This is presently open. On the one side the known characterizations of ultrahomogeneous structures were tested for existence of Ramsey lifts and they confirm the conjecture. But so far this list contain, like for undirected graphs, only variants of known theorems. The central question of this paper can be then formulated.

Question 1.1 *Given a class of structures $\mathcal{K}$ does there exists a finitary lift of $\mathcal{K}$ which is a Ramsey class?*

A necessary condition for validity of Question 1.1 is ω-categoricity and finiteness of the algebraic closure of $\mathcal{K}$.

Our main theorem can be used to establish the validity of Question 1.1 for many new classes of relational structures. The main result stated bellow takes the form of an implication: having proved that certain class $\mathcal{K}$ is Ramsey we can also prove that certain other class $\mathcal{L}$ is Ramsey. The class $\mathcal{L}$ may be much more complex and restrictive than $\mathcal{K}$. Consequently, this result generalizes many results obtained earlier and brings a systematic approach to proving Ramsey property of new classes. This result is obtained by Partite Construction combined with new tools (particularly Partite Lemma for structures with closure). Presently this seems to be the strongest tool for producing new Ramsey classes. Particularly the main result implies Ramsey property of many monotone classes with bounded algebraic closure (such classes were shown to have ω-categorical lift in [1]): for classes with unary algebraic closure the existence of Ramsey lift follows in full generality and we also verified that our method can be applied to known examples of non-unary algebraic closure. This paper uses the experience gained in giving combinatorial proof of [1] and is a far reaching generalization of recent [2], where we found the first Ramsey class (so called bowtie-free graphs) with a non-trivial algebraic closure.

2 Preliminaries

A *structure* $\mathbf{A}$ is a pair $(A, (R^i_{\mathbf{A}}; i \in I))$ where $R^i_{\mathbf{A}} \subseteq A^{\delta_i}$ (i.e. $R^i_{\mathbf{A}}$ is a δ_i-ary relation on A). The finite family $(\delta_i; i \in I)$ is called the *type* Δ. The pair (I, Δ) is called *the language* L. The language is usually fixed and understood from the context. If set A is finite we call $\mathbf{A}$ *finite structure*. A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R^i_{\mathbf{B}}; i \in I))$ is a mapping $f : A \to B$ satisfying for every $i \in I$ the implication $(x_1, x_2, \ldots, x_{\delta_i}) \in R^i_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R^i_{\mathbf{B}}$. If f is 1–1 and these implications are equivalences, then f is called an *embedding*. The class of all (countable) relational structures of type Δ will be denoted by $\mathrm{Rel}(\Delta)$. We can also define structures as models of a language L but for our purposes we need to be little more explicite.

Let $\Delta' = (\delta'_i; i \in I')$ be a type containing type Δ. (That is $I \subseteq I'$ and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{X} \in \mathrm{Rel}(\Delta')$ may be viewed as a structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}; i \in I)) \in \mathrm{Rel}(\Delta)$ together with some additional relations for $i \in I' \setminus I$. We will thus also write $\mathbf{X} = (A, (R^i_{\mathbf{A}}; i \in I), (R^i_{\mathbf{X}}; i \in I' \setminus I))$. We call $\mathbf{X}$ a *lift* of $\mathbf{A}$. Note that a lift is also in the model-theoretic setting called an *expansion* (as we are expanding our relational language by new relations).

Given structure $\mathbf{A}$, relation $R^i_{\mathbf{A}}$ of arity n the *$R^i_{\mathbf{A}}$-out-degree of a k-tuple* $(v_1, v_2, \ldots, v_k)$ is the number of $(n-k)$-tuples $(v_{k+1}, v_{k+2}, \ldots, v_n)$ such that $(v_1, v_2, \ldots, v_n) \in R^i_{\mathbf{A}}$.

Let $\mathcal{F}$ be a family of finite structures. By $\mathrm{Forb_h}(\mathcal{F})$ we denote the class of all finite or countable structures $\mathbf{A}$ such that there is no homomorphism from any $\mathbf{F} \in \mathcal{F}$ to $\mathbf{A}$.

3 Main Result

Because Ramsey classes consists of rigid structures, we make the order an explicit part of the language. Our language will thus always contain at least one binary relation $R^{\leq}$. Structure $\mathbf{A}$ is *ordered* if relation $R^{\leq}$ forms a complete order on A.

Next we need to formalize our notion of closure. Informally the closure of a set S in structure $\mathbf{A}$ is a set of vertices of $\mathbf{A}$ connected to S in a special way which does not permit them to be duplicated. We however consider very restricted notion of this concept:

Definition 3.1 A *closure description* $\mathcal{C}$ is a set pairs $(R^{C_i}, \mathbf{R}_i)$ where R^{C_i} is relation of arity n and $\mathbf{R}_i$ is an ordered structure on at most $n-1$ vertices. We will refer to relations R^{C_i}, as to *closure edges* and to structures $\mathbf{R}_i$ as the *roots of the closures*.

We say that structure $\mathbf{A}$ is $\mathcal{C}$-*closed* if for every pair $(R^{C_i}, \mathbf{R}_i) \in \mathcal{C}$ it holds that the $R^{C_i}_{\mathbf{A}}$-out-degree of an $|R_i|$-tuple $\boldsymbol{r}$ (of vertices of $\mathbf{A}$) is 1 if and only if there is an embedding from $\mathbf{R}_i$ to $\boldsymbol{r}$ and 0 otherwise.

We say that $\mathbf{A}$ is $\mathcal{C}$-*semi-closed* if for every pair it holds that the $R^{C_i}_{\mathbf{A}}$-out-degree of a $|R_i|$-tuple $\boldsymbol{r}$ is:

(i) 1 if $|R_i|=1$ and there is an embedding from $\mathbf{R}_i$ to $\boldsymbol{r}$,

(ii) at most 1 if there is an embedding from $\mathbf{R}_i$ to $\boldsymbol{r}$, and,

(iii) 0 otherwise.

Remark 3.2 For amalgamation classes of ordered structures this definition of closure is equivalent with model-theoretic definition of the algebraic closure in the Fraïssé limit of the class. This follows from fact that the closure edges are definable in the structure and thus their equivalent need to be already present in the language.

Let $\mathcal{F}$ and $\mathcal{R}$ be classes of finite structures. We say that $\mathcal{F}$ *is locally finite in* $\mathcal{R}$ if for every $\mathbf{A} \in \mathcal{R}$ there is only finitely many structures $\mathbf{F} \in \mathcal{F}$ such that there is a homomorphism $\mathbf{F} \to \mathbf{A}$.

Definition 3.3 Let $\mathcal{R}$ be a Ramsey class, $\mathcal{F}$ be a (possibly infinite) family of finite connected ordered structures, and, $\mathcal{C}$ a closure description. We say that countable class $\mathcal{K}$ is $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-*multiamalgamation class* if:

(i) $\mathcal{K}$ is a subclass of the class of (not necessarily all) $\mathcal{C}$-closed structures in $\mathcal{R} \cap \mathrm{Forb_h}(\mathcal{F})$.

(ii) **Regularity of $\mathcal{F}$:** there is ω-categorical universal object in $\mathrm{Forb_h}(\mathcal{F})$.

(iii) **Local finiteness:** $\mathcal{F}$ is locally finite in $\mathcal{R}$.

(iv) **Completetion property:** Let $\mathbf{B}$ be structure from $\mathcal{K}$, $\mathbf{C}$ be $\mathcal{C}$-semi-closed structure with homomorphism to some structure in $\mathcal{R} \cap \mathrm{Forb_h}(\mathcal{F})$ such that every vertex of $\mathbf{C}$ as well as every tuple in every relation of $\mathbf{C}$ is contained in a copy of $\mathbf{B}$. Then there exists $\overline{\mathbf{C}} \in \mathcal{K}$ and a homomorphism $h : \mathbf{C} \to \overline{\mathbf{C}}$ such that h is an embedding on every copy of $\mathbf{B}$ in $\mathbf{C}$ to $\overline{\mathbf{C}}$.

Structure $\mathbf{A}$ is *connected* if its vertex set can not be partitioned into two parts A_1 and A_2 in a way that every tuple in every relation of $\mathbf{A}$ (with exception of the order $R^{\leq}_{\mathbf{A}}$) is fully contained either in A_1 or A_2.

With these concepts we can state our main result compactly as:

Theorem 3.4 *Every $(\mathcal{R}, \mathcal{F}, \mathcal{C})$-multiamalgamation class $\mathcal{K}$ has a Ramsey lift.*

Explicitly: There exists class $\mathcal{L}$ of lifts of structures in $\mathcal{K}$ such that for every pair of structures $\mathbf{A}, \mathbf{B}$ in $\mathcal{L}$ there exists a structure $\mathbf{C} \in \mathcal{L}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Remark 3.5 While our notion of closure is as powerful as algebraic closure for amalgamation classes, applying our result on classes with ω-categorical universal structure needs extra care. The closure edges are typically not explicitely present in the language and their existence is implied by a particular embedding of some structure $\mathbf{P}$. For example, in the case of bow-tie free graphs the closure of a vertex v basically binds vertex v with (some) vertices of triangles containing v, see [2]. This easily translate to binary closure edges in a lifted language that follow the existing edges of the triangles and unary relations describing the type of closure of a vertex. Because of the special case for arity 1 in the definition of semi-closed structures, it is easy to show that all closures of vertices in $\mathbf{C}$ given by Theorem 3.4 are consistent (they are copies of closures of vertices in $\mathbf{B}$).

If relational structure in binary language have a non-unary closure one however need to add explicit relations of higher arities representing the closure. For non-unary closures our construction provide only limited means to prevent introduction of new embeddings of $\mathbf{P}$ and thus to apply our result one needs to carefully verify the completetion property.

The proof of Theorem 3.4 constructs a homogenizing lift with relations representing pieces of structures in $\mathcal{F}$ by technique similar to [3]. The main tool to show Ramsey property is a variant of the the *Partite Construction* developed by Nešetřil-Rödl. In a way our result is further evidence for the surprising effectivity of Partite Construction is the structural Ramsey Theory. Proving our result, we however need to modify the Partite Construction to preserve closure. This was done, for a first time, in [2] and perhaps surprisingly we extend the techniques for non-unary closures. Finally the forbidden homomorphic images are avoided by iterating the Partite Construction in similar sense as in [9]. Full proof will appear in [4].

4 Examples

We believe the above result generalizes all proofs of Ramsey property via Partite Constructions. For example by forbidding regular families $\mathcal{F}$ one can show Ramsey property of metric spaces, ultra-metric spaces, or partial orders. By defining closure one can show Ramsey property of bow-tie-free graphs and classes where Ramsey lift needs additional closure operator, like the semi-generic tournament. We can also show many new classes, including several classes of metric graphs, m-ary functions, and various classes with ternary relations.

In brief, two main directions motivated this paper: We wanted to prove [1] in a combinatorial way. This more explicite proof started to consider forbidden homomorphisms (as explained in [3]) and this then extended to nontrivial algebraic closure in [2]. The second motivation was to extend forbidden homomorphism theorems to some infinite families. In both of these directions this was a successful project as indicated by the main result of this paper.

We give three (admittedly easy) examples to show the method how the Theorem 3.4 can be applied. To get the initial Ramsey class we use the following classical result (an ordered structure $\mathbf{A}$ is *complete* if for every pair of vertices u, w there is tuple $\boldsymbol{v} \in R^i_{\mathbf{A}}$ where i is not $\leq$ such that t contains both u and w):

Theorem 4.1 ([12]) *Let L be a finite relational language and $\mathcal{E}$ be a set of complete ordered L-structures. Let $\mathcal{K}$ be the class of all finite ordered L-structures $\mathbf{A}$ such that $\mathbf{A}$ contain no $\mathbf{E} \in \mathcal{E}$ as an induced substructure. Then the class $\mathcal{K}$ is a Ramsey class.*

4.1 Partial orders with linear extensions

The language of partial orders, in our setting, has binary relations $R^{\prec}$ representing the partial order (we do not include loops in $R^{\prec}$), $R^{\leq}$ representing the linear order, and $R^{\perp}$ denoting that two vertices are not comparable. $R^{\perp}$ is added to make structures complete. By application of Theorem 4.1 we first obtain a Ramsey property of the class $\mathcal{A}$ of all acyclic graphs where oriented edges are represented by $R^{\prec}$ and there is no pair of vertices in both $R^{\prec}$ and $R^{\perp}$. We require the relation $R^{\prec}$ to be subrelation of $R^{\leq}$ so the linear order $R^{\leq}$ forms a linear extension of $R^{\prec}$. Then Ramsey property of partial orders is a modification of Ramseyness of acyclic graphs (as noted in [10]). We formulate this to fit our main theorem:

We put the family of forbidden homomorphic images, $\mathcal{F}_C$, to be a set of all quasi cycles. (A quasi cycle of length n is a structure on vertices $v_1, v_2, \ldots, v_n$ where $(v_1, v_n) \in R^{\perp}$ and $(v_i, v_{i+1}) \in R^{\prec}$, $1 \leq i < n$). Next we verify that the set of all partial orders with linear extension in this language is $(\mathcal{A}, \mathcal{F}_C, \emptyset)$-multiamalgamation class: Regularity of $\mathcal{F}_C$ follows from the existence of generic partial order. Completetion property follows from fact that every acyclic graph in $\mathcal{A}$ not containing any quasi cycle can be completed to a partial order. The critical is local finiteness. While we forbid infinite family of quasi cycles, for every fixed structure $\mathbf{A} \in \mathcal{A}$ there is only finitely many cycles $\mathbf{F} \in \mathcal{F}$ with homomorphic image in $\mathbf{A}$. This is because the oriented path in a quasi cycle must be mapped to a oriented path of same length and thus quasi cycles containing more vertices than $|A|$ have no image in $\mathbf{A}$.

We shall stress that the local finiteness is not only a technical limitation following from the structure of the proof of Theorem 3.4. It is possible to show that the linear extension is definable in every Ramsey lift of the class of all partial orders.

4.2 Multipartite graphs

The n-partite graphs can be shown to be Ramsey by adding an unary relation denoting the individual parts. The forbidden edge within a single partition can be then described by a complete structure and the Ramsey property follows by Theorem 4.1. If we want to add only finitely many relations then the existence of such a lift is not obvious when n is not bounded and the parts are denoted by a binary function representing the equivalence class. We need a different lift that adds a closure. We add a special (closure) vertex that that is used as a unique representative of each equivalence class:

Our language use binary relation R^E for edges, unary relation R^S to denote the special vertices and binary relation R^C representing the unary closure. By Theorem 4.1 we build Ramsey class $\mathcal{R}_P$ in this language. We choose complete forbidden subgraphs to describe the following: 1. force R^E to be symmetric without loops; 2. allow edges only in between non-special vertices; 3. make R^C to be the set of all oriented edges always pointing from nonspecial vertex to the special one.

The closure description $\mathcal{C}_P$ consist of single pair $(R^C, \mathbf{R})$ where $\mathbf{R}$ is structure containing one vertex that is not in relation R^S. We put $(r,s) \in R^{\leq}_{\mathbf{P}}$, $(r,s) \in R^C_{\mathbf{P}}$, and $(s) \in R^S_{\mathbf{P}}$. This represents the fact that every vertex belongs to one partition. Finally we put into $\mathcal{F}_P$ a triangle containing one special vertex and two normal vertices in the same partition connected by an edge (to represent the fact that there are no edges within one partition). It is easy to verify that the class of multipartite graphs with closures is $(\mathcal{R}_P, \mathcal{C}_P, \mathcal{F}_P)$-multiamalgamation class.

This technique can be easily used to construct Ramsey lifts of structures that define an equivalence with infinitely many classes. See, for example, the Ramsey lift of the age of the semigeneric tournament [5] or of the class of bowtie-free graphs [2].

4.3 Binary functions

While the structures with unary closure can be treated easily by Theorem 3.4 for binary closure we need more care. We illustrate this here on perhaps simplest case. We consider ordered structures in language with one binary function assigning every pair of two distinct vertices a vertex: structure $\mathbf{A}$ is triple $(A, R^{\leq}_{\mathbf{A}}, f_{\mathbf{A}})$ where $R^{\leq}_{\mathbf{A}}$ is a linear order of A and $f_{\mathbf{A}}$ is a function $\{(a,b) : a, b \in A, a \neq b\} \to A$. Substructures are induced by $\mathbf{A}$ only on those subsets of A which are closed with respect to function $f_{\mathbf{A}}$. Because our relational structures have no function symbols, we will equivalently interpret those as relational ordered structures in language $L_{\mathcal{B}}$ with relation $R^{\leq}$ and ternary relation R^C, where $(a,b,c) \in R^C_{\mathbf{A}}$ if and only if $f_{\mathbf{A}}(a,b) = c$. Denote by $\mathcal{B}$ the class of all such interpretations of finite structures $(A, R^{\leq}_{\mathbf{A}}, f_{\mathbf{A}})$. To show that $\mathcal{B}$ has Ramsey expansion we apply Theorem 3.4. The closure description $\mathcal{C}_B$ has one pair $(R^C, \mathbf{R})$ where R^C is a discrete structure on two vertices. By application of Theorem 4.1 we obtain Ramsey class $\mathcal{R}_B$ of structures in language $L_{\mathcal{B}}$ where tuples in R^C contain no duplicated vertices. It is easy to see that $\mathcal{B}$ is $(\mathcal{R}_B, \mathcal{C}_B, \emptyset)$-multiamalgamation class because the semi-closed structures can be completed in a free way.

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