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Sudoku Rectangle Completion

(Extended Abstract)

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Abstract

Over the last decade, Sudoku, a combinatorial number-placement puzzle, has become a favorite pastimes of many all around the world. In this puzzle, the task is to complete a partially filled 9×9 square with numbers 1 through 9, subject to the constraint that each number must appear once in each row, each column, and each of the nine 3×3 blocks. Sudoku squares can be considered a subclass of the well-studied class of Latin squares. In this paper, we study natural extensions of a classical result on Latin square completion to Sudoku squares. Furthermore, we use the procedure developed in the proof to obtain asymptotic bounds on the number of Sudoku squares of order n.

Keywords: Sudoku squares, Latin squares, Number of Sudoku squares, Critical sets in Latin squares.

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1 Introduction and preliminaries

A Latin square is an $n \times n$ matrix with entries in $1, \ldots, n$ such that each of the numbers 1 to n appears exactly once in each row and in each column. Latin squares are heavily studied combinatorial objects that date back to the time of Euler and probably earlier. A variant of the notion of Latin squares has recently surfaced in the form of a number-placement puzzle called Sudoku. In this puzzle, the task is to complete a partially filled 9×9 square with numbers 1 through 9 such that in addition to the Latin square conditions, each number appears exactly once in each of the nine 3×3 blocks. This puzzle was popularized in 1986 by the Japanese puzzle company Nikoli and became an international hit in the 2000s, although similar puzzles have appeared in various publications around the world since late 19th century. The emergence of this puzzle has generated a surge of interest in the mathematical properties of Sudoku squares [6].

In this paper, we study a Sudoku rectangle completion problem similar to a classical result of M. Hall on Latin rectangle completion. We start with the formal definition of the main notions used in this paper. Definitions and notations not given here may be found in standard combinatorics and graph theory textbooks such as [2] and [12].

A Latin square of order n is an $n \times n$ matrix with entries from $[n] = \{1, \ldots, n\}$ that satisfies the following two conditions:

- **row condition**: every element in $[n]$ appears at most once in each row.
- **column condition**: every element in $[n]$ appears at most once in each column.

When $n = k^2$ for an integer k, for every $i, j \in [k]$, the (i, j)th block of an $n \times n$ matrix M is defined as the set of entries with coordinates in $((i-1)k + x, (j-1)k + y)$ for $x, y \in [k]$. We say that (i, j) are the coordinates of this block. These blocks partition the set of entries in M into k^2 submatrices, each of size $k \times k$ and therefore containing n entries. The ith row block of M is the union of the blocks at coordinates (i, j) for $j \in [k]$. Similarly, the jth column block of M is the union of the blocks at coordinates (i, j) for $i \in [k]$. A Sudoku square of order $n = k^2$ is an $n \times n$ matrix that in addition to the row and column conditions above, satisfies the following condition:

- **block condition**: every element in $[n]$ appears at most once in each block.

A partial Latin (Sudoku) square of order n is an $n \times n$ matrix with entries from $[n] \cup \{*\}$ (with $*$ representing an *empty* entry) that satisfies the row and column (row, column, and block) conditions. A partial Latin (Sudoku) square P_2 is an extension of a partial Latin (Sudoku) square P_1 if they have the same order and for every $(i, j) \in [n]^2$, if $P_1(i, j) \neq *$, then $P_1(i, j) = P_2(i, j)$. For $m < n$, an $m \times n$ Latin (Sudoku) rectangle is an $n \times n$ partial Latin (Sudoku) square in which all cells in the first m rows are filled and all remaining cells are empty. More generally, for every $p, q \leq n$, a (p, q, n)-Latin (Sudoku) rectangle is an $n \times n$ partial Latin (Sudoku) square in which all cells in the intersection of the first p rows and the first q columns are filled and all remaining cells are empty.

For Latin rectangles there is a well-known theorem of Marshal Hall [5] that states: Every $m \times n$ Latin rectangle can be extended to an $n \times n$ Latin square. This theorem is proved by using the classical matching theorem of Philip Hall. A natural question is whether the similar statement holds for Sudoku rectangles. In this paper, we study this question, and will show that, perhaps surprisingly, the answer depends on the value of m.

For $n = 9$ (the regular Sudoku), this question was answered by Kanaana and Ravikumar [7]. They showed that for all values of m except $m = 5$, an $m \times 9$ Sudoku rectangle can always be completed to a Sudoku square. For $m = 5$, this is not always possible. For example, see the 5×9 Sudoku rectangle in Figure 1. Note that none of the numbers $1, \ldots, 9$ can be placed in the square marked with a star.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
8	3	2	5	6	1	9	4	7
9	6	5	8	4	7	2	3	1
$\star$								

Fig. 1. A 5×9 Sudoku rectangle with no valid completion

For general n, the only previous result on completability of Sudoku rectangle is the following theorem proved by Kanaana and Ravikumar [7].

Theorem A ([7]) *Assume $n = k^2$ and $n - k \leq m < n$. Then every $m \times n$ Sudoku rectangle can be completed to a Sudoku square.*

In this paper, we prove two more sufficient conditions for m, under which every $m \times n$ Sudoku rectangle is completable to an $n \times n$ Sudoku square (Section 2). These results are proved using a two-stage procedure for completing a Sudoku rectangle, where generalized matching and bipartite graph edge coloring is used within the two stages. We will show in Section 3 that the union of these three conditions is a full characterization of the values of m for which every $m \times n$ Sudoku rectangle is completable. An important ingredient of this proof is a constructive lemma that shows how to generalize a non-completable $m \times k$ Sudoku rectangle to a non-completable $m \times n$ Sudoku rectangle. Sections 4 and 5 are devoted to two corollaries of our characterization. We observe in Section 4, that the procedure used to prove our sufficient conditions gives an efficient algorithm for deciding whether a given $m \times n$ rectangle is completable and find a valid completion. Furthermore, in Section 5 we use this procedure combined with theorems of Minc and Van der Waerden, to prove asymptotic bounds on the number of Sudoku squares of order n.

Related work.

Recently quite a few papers have appeared on the mathematical questions raised for Sudoku squares similar to Latin squares. For example in [4] the number of Sudoku squares is discussed, while some results about the sets of mutually orthogonal Sudoku squares are given in [10]. Concepts of "critical (or defining) sets" i.e. "the minimum Sudoku problem" and "Sudoku trades (or detection of unavoidable sets in Sudoku)" are studied in [8], [9], [13], and [3]. In [1] Sudoku is considered as a special case of Gerechte designs and they introduce some interesting mathematical problems about them. The complexity and completeness of finding solution for Sudoku is investigated in [11]. List coloring of Latin and Sudoku graphs is studied in [6] which has an extensive number of references on Sudoku. In [7] the problem of completing Sudoku rectangles is studied.

2 Sufficient Conditions

In Theorem A it is shown that if the Sudoku rectangle only leaves parts of the last row block incomplete, then it can be completed to a Sudoku square. In this section, we prove two theorems, showing that for certain other values of m, an $m \times n$ Sudoku rectangle can always be completed to a Sudoku square. We start with the next result which shows that if the Sudoku rectangle consists of a number of full row blocks and a number of empty row blocks (i.e., no row block is partially filled), then it can always be completed to a Sudoku square.

Theorem 2.1 *Assume $n = k^2$ and $m = lk$ for some $l < k$. Then every $m \times n$ Sudoku rectangle R can be completed to a Sudoku square.*

The detailed proof of this theorem, as well as the proofs of other results in this paper, are omitted here due to lack of space. The high-level idea of the proof is as follows: We prove that we can complete the rectangle R row-block by row-block. To do this, we need to show that R can be extended to a $k(l+1) \times n$ Sudoku rectangle. This is done in two steps: First, we fill the elements in the $(l+1)$st row block in such a way that column and block conditions are satisfied. In the second step, we permute the elements in each column of the $(l+1)$st row block in such a way that the resulting configuration satisfies the row condition as well. Note that column and block conditions will stay satisfied after such a permutation, and therefore, the resulting configuration is a valid Sudoku extension of R.

For the first step, for each block B_d in the $(l+1)$st row block, we formulate an assignment problem as follows: On one side of the assignment problem, we have the numbers $1, \ldots, n$. On the other side, we have the k columns $c_1, c_2, \ldots, c_k$ of the block. A number i can be assigned to a column j, if i does not appear in column j in R. We can use generalizations of Hall's theorem or network flow arguments to show that there is a k-to-1 assignment of numbers to columns in this assignment problem. Putting these assignments together for all blocks B_d of the $(l+1)$st row block, we obtain an assignment of the n numbers to the n columns such that:

- each number is assigned to precisely k columns, one from each block;
- each column has precisely k numbers assigned to it; and
- no number is assigned to a column where it appears in one of the cells of the column above that.

Let us call this assignment M. The second step is to take the assignment M and specify in which of the k rows of the $(l+1)$st row block each number must go. We construct a bipartite graph as follows: On one side, we have the numbers $1, \ldots, n$, and on the other side, we have the n columns of the $(l+1)$st row block. There is an edge between number i and column j, if i is assigned to j under the assignment M. By the properties of M, this is a bipartite k-regular graph. By König's theorem, this graph has an edge-coloring with k colors. Let the colors be denoted by $1, \ldots, k$. We now complete the $(l+1)$st row block as follows: For each edge (i, j) colored with color c, we place the number i on the cth row of the row block and jth column. It is easy to see that with the addition of this row-block all the Sudoku conditions are still satisfied:

There is no repeated element in any row, since the coloring is a proper edge coloring; there is no repeated element in each column by the third property of the assignment M; and there is no repeated element in each block by the first property of the assignment M.

The next theorem gives another sufficient condition for completability of a Sudoku rectangle. Proof (omitted here) follows the same structure as in the above argument, although the arguments for the existence of solutions to the underlying assignment problems are more involved.

Theorem 2.2 *Assume $n = k^2$ and $m = lk + r$ for some $l < k$ and $0 \leq r < k$. Then an $m \times n$ Sudoku rectangle R can always be completed to a Sudoku square, if $(k - r)(k - l) \geq lk$.*

3 A Complete Characterization

In this section, we give some Sudoku rectangle constructions to prove that the union of the three sufficient conditions given in Section 2 and Theorem A is also necessary.

Theorem 3.1 *Assume $n = k^2$ and $m = lk + r$ for some $l < k$ and $0 \leq r < k$. Every $m \times n$ Sudoku rectangle R can always be completed to a Sudoku square, if and only if at least one of the following conditions hold:*

- $l = k - 1$,
- $r = 0$, *or*
- $(k - r)(k - l) \geq lk$.

Sufficiency follows from the theorems in Section 2 and Theorem A. To prove necessity, we need to construct an $m \times n$ Sudoku rectangle that is not completable for any value of m that does not satisfy the conditions of the theorem. The main tool we use in this construction is the following extension of Theorem 2.1, which proves that any (m, k, n)-Sudoku rectangle can be extended to an $m \times n$ Sudoku rectangle. This means that to construct an $m \times n$ Sudoku rectangle that is not completable, it is enough to construct its first column block, i.e., an (m, k, n)-Sudoku rectangle that is not completable.

Lemma 3.2 *For every $n = k^2$, $m \leq n$, every (m, k, n)-Sudoku rectangle R can be extended to an (m, n, n)-Sudoku rectangle.*

The proof of the above lemma as well as the construction of non-completable (m, k, n)-Sudoku rectangle is left to the full version of the paper.

4 Algorithms for Sudoku Rectangle Completion

The two-stage procedure used in the proof of Theorem 2.2 is essentially a polynomial-time algoritm for completing a Sudoku rectangle into a Sudoku square. Furthermore, we note that the same procedure applies for every value of m, and not just for those for which Theorem 2.2 guarantees the existence of a Sudoku square. This is a simple observation, and follows from the fact that if the given Sudoku rectangle R has a valid extension, then this extension gives a perfect 1-to-$(k-r)$ matching in the graph constructed in the first stage. On the other hand, if R is not completable, then the graph constructed in the first stage does not have any perfect 1-to-$(k-r)$ matching, since if it did, this matching can be turned into an extension to a Sudoku square using the method outlined in the proof of Theorem 2.2. Therefore, we have the following result.

Theorem 4.1 *There is a polynomial time algorithm that given an $m \times n$ Sudoku rectangle R, finds a completion of R to a Sudoku square, or decides that R is not completable to a Sudoku square.*

An interesting follow-up question is whether the above result can be generalized to the case where R is a general partial Sudoku square or when R is a (p, q, n)-Sudoku rectangle. We conjecture that at least in the former case, the problem is NP-complete.

5 Asymptotics of the Number of Sudoku Squares

Let $\mathrm{Sud}(n)$ denote the number of Sudoku squares of order n. In this section, we give a tight asymptotic bound on $(\mathrm{Sud}(n))^{1/n^2}$. Our proof uses the Sudoku rectangle completion procedure of the Section 3 as well as classical bounds on the permanent of matrices. Our result is summarized in the following theorem.

Theorem 5.1 *We have $(\mathrm{Sud}(n))^{1/n^2} \sim e^{-3}n$ as $n \to \infty$.*

This should be contrasted with a similar result for Latin squares (see, for example, [12, Theorem 17.3]), which shows that $(L(n))^{1/n^2} \sim e^{-2}n$ as $n \to \infty$, where $L(n)$ is the number of Latin squares of order n. In other words,

Corollary 5.2 *The fraction of the Latin squares of order n that are Sudoku squares is $(\frac{1}{e} + o(1))^{n^2}$.*

The proof of Theorem 5.1 (omitted here) uses two classical results about the permanent of matrices: The Van der Waerden conjecture (proved independently by Falikman and Egoritsjev), and Minc's conjecture (proved by Brégman) that give lower and upper bounds on the number of solutions to an assignment problem (See [12]). We use the Sudoku rectangle completion procedure of Section 3 to build a Sudoku square row-block by row-block, and in each step bound the number of ways each row-block can be completed using the above theorems. The result follows after a careful calculation using Stirling's formula.

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