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Local colourings and monochromatic partitions in complete bipartite graphs

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Abstract

We show that for any 2-local colouring of the edges of a complete bipartite graph, its vertices can be covered with at most 3 disjoint monochromatic paths. And, we can cover almost all vertices of any complete or complete bipartite r-locally coloured graph with $O(r^2)$ disjoint monochromatic cycles.

Keywords: monochromatic cycle partition, monochromatic path partition, local colouring.

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1 Introduction

1.1 History of monochromatic partitions

The problem of partitioning a graph into few monochromatic paths or cycles, first formulated explicitly in the beginning of the 80's [6], has lately received a fair amount of attention. Its origin lies in Ramsey theory and its subject are complete graphs (later substituted with other types of graphs), whose edges are coloured with r colours. Call such a colouring an r-colouring; note that this need not be a proper edge-colouring. The challenge is now to find a small number of disjoint monochromatic paths, which together cover the vertex set of the underlying graph. Or, instead of disjoint monochromatic paths, we might ask for disjoint monochromatic cycles. Here, single vertices as well as single edges count as cycles. Such a cover is called a monochromatic path partition, or a monochromatic cycle partition, respectively. It is not difficult to construct r-colourings that do not allow for partitions into less than r paths, or cycles. For instance, take vertex sets $V_1, \ldots, V_r$ with $|V_i| = 2^i$, and for $i \leq j$ give all V_i–V_j edges colour i.

It has been long known that for $r = 2$, any r-coloured complete graph K_n has a partition into two disjoint paths, regardless of the size of n [5]. Moreover, these paths have different colours. An extension of this fact, namely that every 2-colouring of K_n has a partition into two monochromatic cycles of different colours was conjectured by Lehel, and verified for large n in [17] and in [1], and for all values of n in [2].

A generalisation of these two results for other values of r was conjectured by Gyárfás, and by Erdős, Gyárfás and Pyber, respectively. More precisely, they conjectured that any r-coloured K_n can be partitioned into r monochromatic paths [7], and even into r monochromatic cycles [4]. The conjecture for cycles was recently disproved by Pokrovskiy [19]. He gave counterexamples for all $r \geq 3$, but also showed that the conjecture for paths is true for $r = 3$. It is further known that any r-coloured K_n can be partitioned into $O(r \log r)$ monochromatic cycles [9]. We remark that Pokrovskiy's counterexamples have partitions into $r+1$ cycles (and one of these is a single vertex), and he conjectures r disjoint cycles suffice for covering all but a constant number $c = c(r)$ of vertices [19].

Monochromatic path/cycle partitions have also been studied for bipartite graphs, mainly for $r = 2$. Interestingly, a certain class of colourings makes a direct extension of the results above impossible. A 2-colouring of $K_{n,n}$ is called a split colouring if there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a properly edge-coloured $K_{2,2}$. It turns out that in the case of a split colouring one might need three monochromatic paths, but otherwise $K_{n,n}$ can be partitioned into two paths of distinct colours [19].

Split colourings can be generalised to more colours, providing a general lower bound of $2r-1$ on the path/cycle partition number for $K_{n,n}$. For $r=3$, this bound is asymptotically correct [14]. For an upper bound, it is known that any r-coloured $K_{n,n}$ can be partitioned into $O(r^2 \log r)$ monochromatic cycles [18]. As a byproduct of Theorem 1 (b) below, we obtain an improvement of this bound to $O(r^2)$.

1.2 Local colourings and our results

We establish new bounds for the size of cycle partitions with respect to local colourings. Local colourings are a natural way to generalise r-colourings, and first appeared in the context of Ramsey theory [8,21]. A colouring of the edges of a graph is said to be r-local if every vertex is adjacent to edges of at most r distinct colours. For instance, a rainbow colouring of the triangle is 2-local.

Recently, Conlon and Stein [3] showed that any r-local colouring of K_n admits a partition into $O(r^2 \log r)$ monochromatic cycles, and, if $r=2$, then two cycles (of different colours) suffice. We improve their bound for complete graphs, and also give a bound for monochromatic cycle partitions in bipartite graphs.

Theorem 1.1 *[15] For every $r \geq 1$ there is an n_0 such that for $n \geq n_0$ the following holds.*

(a) If K_n is r-locally coloured, then all its vertices can be covered with at most $2r^2$ disjoint monochromatic cycles.

(b) If $K_{n,n}$ is r-locally coloured, then all its vertices can be covered with at most $4r^2$ disjoint monochromatic cycles.

The proof of Theorem 1.1 is described in Section 2. We do not believe our results are best possible, but suspect that in both cases (K_n and $K_{n,n}$), the number of cycles needed should be linear in r.

Conjecture 1.2 *There is a c such that for every r, any r-local colouring of K_n or of $K_{n,n}$ admits a covering with cr disjoint cycles.*

Our second result treats the case $r=2$:

Theorem 1.3 *[15] Let the edges of $K_{n,n}$ be coloured 2-locally. Then $K_{n,n}$ can be partitioned into 3 or less monochromatic paths.*

The proof of Theorem 1.2 is purely combinatorial. We first break down the colouring to three possible shapes. It is then shown that for each of the shapes $K_{n,n}$ can be partitioned into 3 or less monochromatic paths. For details, see [15].

2 Proof of Theorem 1.1

In this section we give an outline of the proof of Theorem 1.1(a). The proof of Theorem 1.1(b) is very similar.

2.1 Part I: Monochromatic connected matchings

We need the notion of monochromatic connected matchings. The use of these matching, together with an application of the regularity lemma (see below) was first used by Łukzak [16] and has by now become a standard approach. A monochromatic connected matching is a matching in a connected component of the graph spanned by the edges of a single colour.

The following lemma plays a key role in the proof.

Lemma 2.1 *[15] If K_n is r-locally edge coloured, then $V(K_n)$ can be covered with at most $r(r+1)/2$ monochromatic connected matchings.*

The proof of Lemma 2.1 uses induction, and the following simple lemma, which itself is proved with a greedy strategy:

Lemma 2.2 *[15] For $k \geq 2$, let the edges of a graph G be coloured k-locally. Suppose there are m monochromatic components that together cover $V(G)$, of colours $c_1, \ldots, c_m$.*
Then there are m vertex-disjoint monochromatic connected matchings $M_1, \ldots, M_m$, of colours $c_1, \ldots, c_m$, such that the inherited colouring of $G - V(\bigcup_{i=1}^{m} M_i)$ is a $(k-1)$-local colouring.

2.2 Part II: Regularity

The aim of this part of the proof is to show the following auxiliary lemma.

Lemma 2.3 *[15] If K_n is r-locally edge coloured, then all but $o(n)$ vertices of K_n can be covered with at most $r(r+1)/2$ monochromatic cycles.*

For our proof of Lemma 2.3, we assume the reader's familiarity with Szemerédi's regularity lemma [20], as described in any standard graph theory textbook. We will use the regularity lemma for edge-coloured graphs, which gives a partition, where almost all pairs are highly regular in all colours. For this, however, it is necessary that the total number of colours in bounded, a condition our host graph, the r-locally coloured K_n, does not a priori fulfill.

So, for our application of the multi-coloured regularity lemma we simply delete all edges of all the colours of lowest density, as these turn out to be few. We then obtain a vertex-partition of K_n, where most pairs are highly regular in all of the colours. The reduced graph R inherits an edge-colouring from the r-local colouring of K_n, by simply giving each pair the colour which has the largest density in the pair. The graph R then turns out to be an almost complete bipartite graph that is r-locally coloured. For details see [15].

We now use a robust version of Lemma 2.1, which permits us to partition almost all vertices of R into $r(r+1)/2$ monochromatic connected matchings. In the subsequent step, we apply a specific case of the blow up lemma (see [10,13,16]) in order to find $r(r+1)/2$ monochromatic cycles, which together form a partition of almost all vertices of K_n, using the monochromatic connected matchings of R as a roadmap. This proves Lemma 2.3.

2.3 Part III: Absorption

For the rest of the proof of Theorem 1.1(a), we combine ideas of [11] and [12] with Lemma 2.3.

In a first step we find a large monochromatic uniform subgraph H of the r-locally coloured complete graph K_n from Theorem 1.1(a). More precisely, H has the property that it remains hamiltonian even if $o(|V(H)|)$ of its vertices are deleted. The size of H is linear in n. The existence of such a subgraph H follows from results of [11,12], see also [14].

Next, we cover almost all vertices of $K_n - V(H)$ with the help of Lemma 2.3, using at most $r(r+1)/2$ vertex-disjoint monochromatic cycles. Denote by S the vertices of $K_n - V(H)$ that are not covered by these cycles.

Observe that by choosing n large enough, we can ensure that the size of S sufficiently small compared to the order of H, to allow for applying the following lemma of [3]:

Lemma 2.4 *Suppose that A and B are vertex sets with $|B| \leq |A|/r^{r+3}$ and the complete bipartite graph between A and B is r-locally coloured. Then all vertices of B can be covered with at most r^2 disjoint monochromatic cycles.*

Apply Lemma 2.4 to the graph induced by the edges between S and H. This gives a cover of the vertices of S with at most r^2 vertex-disjoint monochromatic cycles, whose union we denote by C. The robust hamiltonicity of H

(i.e. the property from above) guarantees that $H - V(C)$ is hamiltonian. As H is monochromatic we can finish by taking one more monochromatic cycle, which covers $H - V(C)$.

In total, we used $r(r+1)/2 + r^2 + 1 \leq 2r^2$ disjoint monochromatic cycles to cover all of K_n. This proves Theorem 1.1(a).

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