Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Classifying partial Latin rectangles

Raúl M. Falcón [1,2]

Department of Applied Mathematics I
University of Seville
Seville, Spain

Rebecca J. Stones [3,4]

College of Computer and Control Engineering
Nankai University
Tianjin, China

Abstract

Isotopisms of the set $\mathcal{R}_{r,s,n}$ of $r \times s$ partial Latin rectangles based on n symbols constitute a finite group that acts on this set by permuting rows, columns and symbols. The number of partial Latin rectangles preserved by this action only depends on the conjugacy classes of these permutations. In this paper, the distribution of the isotopism group into conjugacy classes is considered in order to determine the distribution of $\mathcal{R}_{r,s,n}$ into isomorphism and isotopism classes, for all $r, s, n \leq 6$.

Keywords: Partial Latin rectangle, autotopism group, conjugacy.

[1] Supported by PAIDI Andalusian group FQM-016.

[2] Email: `rafalgan@us.es`

[3] Supported by her NSF China Research Fellowship for International Young Scientists (grant number: 11450110409).

[4] Email: `rebecca.stones82@gmail.com`

1 Introduction

An $r \times s$ *partial Latin rectangle* based on a set of n symbols is an $r \times s$ array $P = (p_{ij})$ in which each cell is either empty or contains one element chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. If $r = s = n$, then P is a *partial Latin square of order* n. Hereafter, $[n] = \{1, \ldots, n\}$ is assumed to be this set of symbols and $\mathcal{R}_{r,s,n}$ denotes the set of $r \times s$ partial Latin rectangles based on $[n]$. The *size* of P is defined as the number of non-empty cells. Its *orthogonal array representation* is the set $O(P) = \{(i, j, p_{ij}) \in [r] \times [s] \times [n]\}$. If there does not exist empty cells and $s = n$, then P is a *Latin rectangle* (a *Latin square* if $r = s = n$).

Let S_m denotes the symmetric group on m elements. The *isotopism group* $S_r \times S_s \times S_n$ constitutes a finite group that acts on the set $\mathcal{R}_{r,s,n}$ by permuting rows, columns and symbols. Let $\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$. It is defined the *isotopic* partial Latin rectangle P^Θ whose orthogonal array representation is $O(P^\Theta) = \{(\alpha(i), \beta(j), \gamma(p_{i,j}))\colon (i, j, p_{ij}) \in O(P)\}$. The triple Θ is said to be an *isotopism* of $\mathcal{R}_{r,s,n}$. If $r = s = n$ and $\alpha = \beta = \gamma$, then Θ is an *isomorphism*. In this case, the symmetric group S_n is identified with the *isomorphism group* of $\mathcal{R}_{n,n,n}$. To be isotopic or isomorphic are equivalence relations among partial Latin rectangles. The stabilizer groups of a partial Latin rectangle by the respective actions of $S_r \times S_s \times S_n$ and S_n constitute its *autotopism* and *automorphism* groups.

The enumeration and classification of partial Latin rectangles is an open question. Even though the number of Latin rectangles in $\mathcal{R}_{r,n,n}$ is currently known [8] for all $r, n \leq 11$, the cardinality of the set $\mathcal{R}_{r,s,n}$ is only known for $r, s, n \leq 4$, which was recently been obtained [3,4] by identifying this set with the affine variety defined by the zero-dimensional radical ideal of polynomials $I_{r,s,n} = \langle\, x_{ijk}(x_{ijk} - 1),\, x_{ijk}x_{i'jk},\, x_{ijk}x_{ij'k},\, x_{ijk}x_{ijk'} \colon i \in [r], j \in [s], k \in [n], i' \in \{i+1, \ldots, r\}, j' \in \{j+1, \ldots, s\}, k' \in \{k+1, \ldots, n\}\,\rangle \subseteq \mathbb{Z}/2\mathbb{Z}[x_{111}, \ldots, x_{rsn}]$. Particularly, every partial Latin rectangle $P = (p_{ij}) \in \mathcal{R}_{r,s,n}$ is uniquely identified with a point $(a_{111}, \ldots, a_{rsn}) \in \{0,1\}^{rsn}$, where $a_{ijk} = 1$ if $p_{ij} = k$ and 0, otherwise. The decomposition of the affine variety $V(I_{r,s,n})$ into finitely many disjoint subsets, each of them being the zeros of a triangular system of polynomials equations, makes possible to determine in Section 2 the number of $r \times s$ partial Latin rectangles based on n symbols, for all $r, s, n \leq 6$.

Even though the distribution of Latin squares into isomorphism and isotopism classes has been determined [7,9] for order up to 11, it is only known [1] the number of isotopism classes of those partial Latin squares $P \in \mathcal{R}_{n,n,n}$ of order $n \leq 6$ that are *critical sets*. This paper deals with the use of conjugacy classes of isotopisms together with Burnside's lemma in order to determine the distribution of $r \times s$ partial Latin rectangles based on $[n]$ into isomorphism and isotopism classes, for all $r, s, n \leq 6$.

2 Enumeration of $\mathcal{R}_{r,s,n}$

Given a positive integer $i \leq r$, let $I^{(i)}_{r,s,n}$ be the subideal in $I_{r,s,n}$ related to the i^{th} row of an $r \times s$ partial Latin rectangle based on $[n]$, that is, $I^{(i)}_{r,s,n} = \langle\, x^2_{ijk} - x_{ijk},\, x_{ijk}x_{ij'k},\, x_{ijk}x_{ijk'} \colon\, j, j' \leq s,\, k, k' \leq n, j < j', k < k' \,\rangle$. Let $\{J_{1,1}, \ldots, J_{1,t}\}$ be a finite set of subideals of $I^{(1)}_{r,s,n}$ whose affine varieties constitute a partition of the affine variety of the ideal $I^{(1)}_{r,s,n}$ and which are generated by the polynomials of t distinct triangular systems of polynomial equations. This finite set can be obtained by applying the algorithm of Moeller and Hillebrand [6,10]. Given $l \leq t$ and $i \in \{2, \ldots, r\}$, let $J_{i,l}$ be the subideal of $I^{(i)}_{r,s,n}$ whose generators coincide with those of the subideal $J_{1,l}$ after replacing the variable x_{1jk} by x_{ijk}, for all $j \leq s$ and $k \leq n$. For each tuple $(t_1, \ldots, t_r) \in [t]^r$, let us define the ideal $K_{t_1,\ldots,t_r} = J_{1,t_1} + \ldots + J_{r,t_r} + J$, where $J = \langle\, x_{ijk}x_{i'jk} \colon\, i, i' \leq r, j \leq s, k \leq n, i < i' \,\rangle$. The set of affine varieties $\{V(K_{t_1,\ldots,t_r}) \colon\, (t_1, \ldots, t_r) \in [t]^r\}$ constitutes a partition of $V(I_{r,s,n})$, whose cardinalities have been explicitly obtained for all $r, s, n \leq 6$ by means of the set of Gröbner bases and Hilbert functions of the corresponding ideals $K_{t_1,\ldots,t_r}$. They coincide with the number of $r \times s$ partial Latin rectangles based on $[n]$ that are given in Table 1.

3 Distribution into isomorphism and isotopism classes

This section deals with the distribution of $r \times s$ partial Latin rectangles based on $[n]$ into isomorphism and isotopism classes. Due to conjugacy of rows, columns and symbols, we focus on the case $r \leq s \leq n$. Let $\mathcal{I}_n$ and $\mathfrak{I}_{r,s,n}$ denote, respectively, the sets of isomorphism and isotopism classes of $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{n,n,n}$. The next result follows straightforward.

Lemma 3.1 *Let r, s and n be three positive integers. If $rs \leq n$, then $|\mathfrak{I}_{r,s,n}| = |\mathfrak{I}_{r,s,rs}|$. Further, if $s \leq n$, then $|\mathfrak{I}_{1,s,n}| = s + 1$.* □

Similarly to (partial) Latin squares [2,3,11], the distribution of partial Latin rectangles into isotopism and isomorphism classes can be determined according to the conjugacy classes of $S_r \times S_s \times S_n$. In this regard, let $\mathcal{R}_\Theta$ denote the set of $r \times s$ partial Latin rectangles based on $[n]$ that have an

r	s	n	$\lvert\mathcal{R}_{r,s,n}\rvert$	r	s	n	$\lvert\mathcal{R}_{r,s,n}\rvert$	r	s	n	$\lvert\mathcal{R}_{r,s,n}\rvert$
1	1	1	2	1	5	6	4051	3	3	5	805366
		2	3		6	6	13327			6	4193269
		3	4	2	2	2	35		4	4	2423521
		4	5			3	121			5	33199561
		5	6			4	325			6	317651473
		6	7			5	731		5	5	890442316
	2	2	7			6	1447			6	15916515301
		3	13		3	3	781		6	6	526905708889
		4	21			4	3601	4	4	4	127545137
		5	31			5	12781			5	4146833121
		6	43			6	37273			6	87136329169
	3	3	34		4	4	28353		5	5	313185347701
		4	73			5	162661			6	14554896138901
		5	136			6	720181		6	6	1474670894380885
		6	229		5	5	1502171	5	5	5	64170718937006
	4	4	209			6	10291951			6	7687297409633551
		5	501		6	6	108694843		6	6	2322817844850428851
		6	1045	3	3	3	11776	6	6	6	2027032853070203981647
	5	5	1546			4	116425				

Table 1
Number of $r \times s$ partial Latin rectangles based on $[n]$, for all $r \leq s \leq n \leq 6$.

isotopism $\Theta \in S_r \times S_s \times S_n$ in its autotopism group. The next result holds.

Lemma 3.2 *Let Θ_1 and Θ_2 be two conjugate isotopisms in $S_r \times S_s \times S_n$. It is verified that $|\mathcal{R}_{\Theta_1}| = |\mathcal{R}_{\Theta_2}|$. Further, the set of isotopism classes of $\mathcal{R}_{\Theta_1}$ coincides with that of $\mathcal{R}_{\Theta_2}$.*

Proof. Let $\Theta_3 \in S_r \times S_s \times S_n$ be such that $\Theta_2 = \Theta_3\Theta_1\Theta_3^{-1}$. The result follows straightforward from the fact that $\mathcal{R}_{\Theta_2} = \{P^{\Theta_3} : \ P \in \mathcal{R}_{\Theta_1}\}$. $\square$

To be conjugate is an equivalence relation among isotopisms in which each conjugacy class is characterized by the common cycle structure of its elements. Recall that the *cycle structure* of a permutation π in the symmetric group S_m is defined as the expression $z_\pi = m^{d_m^\pi} \ldots 1^{d_1^\pi}$, where d_i^π is the number of cycles of length i in the unique cycle decomposition of the permutation π. In practice, it is only written those terms $i^{d_i^\pi}$ for which $d_i^\pi > 0$. The *cycle structure* of an isotopism $(\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ is then defined as the triple $(z_\alpha, z_\beta, z_\gamma)$ formed by the respective cycle structures of α, β and γ. Thus, for instance, the cycle structure of the isotopism $((1234), (12)(3)(45), (12)(345)(6)) \in S_4 \times S_5 \times S_6$ is $(4, 2^21, 321)$. Let $\mathcal{CS}_m$ denote the set of cycle structures of the symmetric group S_m. Given a cycle structure $z \in \mathcal{CS}_m$, let d_i^z denote the value of d_i^π for all permutation $\pi \in S_m$ of cycle structure z. Similarly to the case of (partial) Latin squares [3,11], the next result describes the cycle structure of any isotopism of $\mathcal{R}_{r,s,n}$.

Lemma 3.3 *A triple $z=(z_1,z_2,z_3)\in\mathcal{CS}_r\times\mathcal{CS}_s\times\mathcal{CS}_n$ is the cycle structure of an isotopism of a non-empty $r\times s$ partial Latin rectangle based on $[n]$ if and only if there exists a triple $(i,j,k)\in[r]\times[s]\times[n]$ such that $\mathrm{lcm}(i,j)=\mathrm{lcm}(i,k)=\mathrm{lcm}(j,k)=\mathrm{lcm}(i,j,k)$ and $d_i^{z_1}\cdot d_j^{z_2}\cdot d_k^{z_3}>0$.* $\square$

Let $\Delta(z_1,z_2,z_3)$ denote the cardinality of the set $\mathcal{R}_\Theta$ for any isotopism Θ of cycle structure $(z_1,z_2,z_3)\in\mathcal{CS}_r\times\mathcal{CS}_s\times\mathcal{CS}_n$. This can be used to determine the distribution of partial Latin rectangles into isotopism and isomorphism classes. Specifically, since the isotopism and the isomorphism groups are finite groups that acts on $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{n,n,n}$, respectively, Burnside's lemma and the number of permutations with a given cycle structure involve:

$$|\mathfrak{I}_{r,s,n}|=\sum_{\substack{\alpha\in S_r\\ \beta\in S_s\\ \gamma\in S_n}}\frac{\Delta(z_\alpha,z_\beta,z_\gamma)}{r!s!n!}=\sum_{\substack{z_1\in\mathcal{CS}_r\\ z_2\in\mathcal{CS}_s\\ z_3\in\mathcal{CS}_n}}\frac{\Delta(z_1,z_2,z_3)}{\prod_{\substack{i\in[r]\\ j\in[s]\\ k\in[n]}}d_i^{z_1}!d_j^{z_2}!d_k^{z_3}!\,i^{d_i^{z_1}}\,j^{d_j^{z_2}}\,k^{d_k^{z_3}}}. \tag{1}$$

$$|\mathcal{I}_n|=\sum_{\pi\in S_n}\frac{\Delta(z_\pi,z_\pi,z_\pi)}{n!}=\sum_{z\in\mathcal{CS}_n}\frac{\Delta(z,z,z)}{\prod_{i\in[n]}d_i^z!i^{d_i^z}}. \tag{2}$$

The next result shows how the set $\mathcal{R}_\Theta$ and its cardinality can explicitly be determined by means of the affine variety defined by a zero-dimensional radical subideal of the ideal I_{rsn}.

Theorem 3.4 *Let $\Theta=(\alpha,\beta,\gamma)$ be an isotopism of $\mathcal{R}_{r,s,n}$. The set $\mathcal{R}_\Theta$ is identified with the affine variety defined by the ideal $I_\Theta=I_{r,s,n}\cup\langle\, x_{ijk}-x_{\alpha(i)\beta(j)\gamma(k)}\colon\ i\in[r],\ j\in[s],\ k\in[n]\,\rangle\subseteq\mathbb{Z}/2\mathbb{Z}[x_{111},\ldots,x_{rsn}]$. Further, $\Delta(z_\alpha,z_\beta,z_\gamma)=|\mathcal{R}_\Theta|=\dim_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}[x_{111},\ldots,x_{rsn}]/I_\Theta)$.*

Proof. Let $(a_{111},\ldots,a_{rsn})\in\{0,1\}^{rsn}$ be a point of the affine variety defined by the ideal I_Θ. Since it is a subideal of $I_{r,s,n}$, this point is uniquely related to the partial Latin rectangle $P=(p_{ij})\in\mathcal{R}_{r,s,n}$ such that $p_{ij}=k$ if $a_{ijk}=1$ and $\emptyset$, otherwise. Further, the binomials $x_{ijk}-x_{\alpha(i)\beta(j)\gamma(k)}$ that have been incorporated in the set of generators of the ideal $I_{r,s,n}$ in order to define the new ideal I_Θ involve our point to satisfy the identity $a_{ijk}=a_{\alpha(i)\beta(j)\gamma(k)}$, for all $(i,j,k)\in[r]\times[s]\times[n]$. This is equivalent to the identity $p_{\alpha(i)\beta(j)}=\gamma(p_{ij})$, for all $(i,j)\in[r]\times[s]$ such that $p_{ij}\in[n]$. It involves the isotopism Θ to be an autotopism of P. $\square$

In order to determine the affine variety in Theorem 3.4, we implemented the procedure *PLRT* in Singular [5], which is available online on the URL http://www.personal.us.es/raufalgan/LS/pls.lib. This has been used to compute the values $\Delta(z_1, z_2, z_3)$ for all $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$ satisfying the hypothesis of Lemma 3.3, being $r, s, n \leq 6$. Expressions (1) and (2) have then be applied to determine the distribution of $\mathcal{R}_{r,s,n}$ into isomorphism and isotopism classes in Tables 2 and 3.

n	$\lvert\mathcal{I}_n\rvert$
1	2
2	20
3	2029
4	5319934
5	534759300182
6	2815323435872410905

Table 2
Distribution of partial Latin squares into isomorphisms classes.

r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$	r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$	r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$	r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$
2	2	2	8	2	4	4	54	3	3	6	325	4	4	6	218558
		3	10			5	70		4	4	839		5	5	914969
		4	11			6	78			5	2227			6	7074338
		5	11		5	5	125			6	3825		6	6	118883849
		6	11			6	166		5	5	11194	5	5	5	37202840
	3	3	20		6	6	292			6	33299			6	742190170
		4	27	3	3	3	81		6	6	177892		6	6	37349106398
		5	29			4	184	4	4	4	9878	6	6	6	5431010366322
		6	30			5	279			5	61955				

Table 3
Distribution of partial Latin rectangles into isotopism classes.

References

[1] Adams, P., R. Bean and A. Khodkar, *A census of critical sets in the Latin squares of order at most six*, Ars Comb. **68** (2003), 203–223.

[2] Falcón, R. M., *Cycle structures of autotopisms of the Latin squares of order up to 11*, Ars Comb. **103** (2012), 239–256.

[3] Falcón, R. M., *The set of autotopisms of partial Latin squares*, Discrete Math. **313** (2013), 1150–1161.

[4] Falcón, R. M., *Enumeration and classification of self-orthogonal partial Latin rectangles by using the polynomial method*, European J. Combin. **48** (2015), 215–223.

[5] Decker, W., G. M. Greuel, G. Pfister and H. Schönemann, Singular *4-0-2 — A computer algebra system for polynomial computations* (2015), URL: http://www.singular.uni-kl.de.

[6] Hillebrand, D., "Triangulierung nulldimensionaler ideale - implementierung und vergleich zweier algorithmen", Master's thesis, Universitaet Dortmund, Fachbereich Mathematik, 1999.

[7] Hulpke, A., P. Kaski and P. R. J. Östergård, *The number of Latin squares of order 11*, Math. Comp. **80** (2011), 1197–1219.

[8] McKay, B. D. and I. M. Wanless, *On the number of Latin squares*, Ann. Comb. **9** (2005), 335–344.

[9] McKay, B. D., A. Meynert and W. Myrvold, *Small Latin Squares, Quasigroups and Loops*, J. Combin. Des. **15** (2007), 98–119.

[10] Möller, H. M., *On decomposing systems of polynomial equations with finitely many solutions*, Appl. Algebra Engrg. Comm. Comput. **4** (1993), 217–230.

[11] Stones, D. S., P. Vojtěchovský and I. M. Wanless, *Cycle structure of autotopisms of quasigroups and Latin squares*, J. Combin. Des. **20** (2012), 227–263.