



Strong edge coloring sparse graphs

Julien Bensmail

LIP (École Normale Supérieure de Lyon), 46 allée d'Italie, 69342 Lyon, France

Marthe Bonamy¹

LIRMM (Université Montpellier), 161 rue Ada, 34095 Montpellier, France

Hervé Hocquard

LaBRI (Université de Bordeaux), 351 cours de la Libération, 33405 Talence, France

Abstract

A strong edge coloring of a graph is a proper edge coloring such that no edge has two incident edges of the same color. Erdős and Nešetřil conjectured in 1989 that $\frac{5}{4}\Delta^2$ colors are always enough for a strong edge coloring, where Δ is the maximum degree of the graph. In the specific case where $\Delta = 4$, we prove this to be true when there is no subgraph with average degree at least $4 - \frac{1}{5}$, and show that fewer colors are necessary when the graph is even sparser.

Keywords: strong edge coloring, strong chromatic index, maximum average degree, discharging method.

¹ This author is partially supported by the ANR Grant EGOS (2012-2015) 12 JS02 002 01.

1 Introduction

Let G be a simple undirected graph. A proper edge coloring of G is *strong* if every color class provides an induced matching of G . The least number of colors in a strong edge coloring of G is the *strong chromatic index* of G , commonly denoted by $\chi'_s(G)$. Since the introduction of this parameter by Fouquet and Jolivet in 1983 [8], the question of determining the exact value of the strong chromatic index of graphs has attracted great attention (see *e.g.* [10,11] for an overview of the topic).

This question is hard to answer even in restricted settings like the class of planar subcubic graphs with no triangle (see *e.g.* [9]). Therefore, many works focus instead on exhibiting upper bounds on the strong chromatic index of particular families of graphs (see again [10,11] for examples of such bounds). Most of these upper bounds are expressed as a function of Δ , where Δ refers from now on to the maximum degree of the graph considered. This is justified by the fact that Δ is a trivial lower bound on χ'_s , and that greedy coloring arguments yield a natural upper bound of $2\Delta^2 - 2\Delta + 1$ on χ'_s .

This upper bound is actually quite naïve, as most graphs can be strongly edge colored using fewer colors. Improving the upper bound involving Δ is at the heart of many investigations on the strong chromatic index. Let us mention those concerning two specific families of graphs: planar graphs and bipartite graphs. In the setting of planar graphs, Theorem 1.1 stands out.

Theorem 1.1 (Faudree *et al.* [7]) *Every planar graph G satisfies $\chi'_s(G) \leq 4\Delta + 4$.*

The upper bound of Theorem 1.1 is correct up to an additive factor, as there is a family of planar graphs with strong chromatic index $4\Delta - 4$ [7]. For more details on recent improvements on the strong chromatic index of planar graphs, see [2]. Regarding bipartite graphs, the goal is the following conjecture.

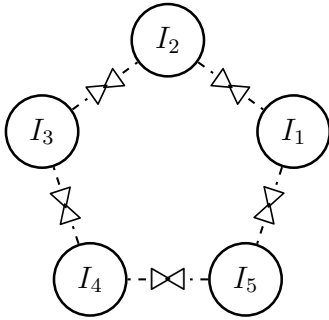
Conjecture 1.2 (Faudree *et al.* [6]) *Every bipartite graph G satisfies $\chi'_s(G) \leq \Delta^2$.*

If correct, this is tight due to complete bipartite graphs of the form $K_{n,n}$. See [3] for an up-to-date survey on the investigations related to Conjecture 1.2. There is also a conjecture in the general case, as follows.

Conjecture 1.3 (Erdős and Nešetřil [5]) *Every graph G satisfies*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \text{otherwise.} \end{cases}$$

If true, the bounds in Conjecture 1.3 are tight, as confirmed by the Erdős-Nešetřil graphs whose construction is described in Figure 1.



- Every I_j is an independent set.
- " $I_j \bowtie I_{j'}$ " means that I_j is complete to $I_{j'}$.
- If $\Delta = 2k$, then $|I_j| = k$.
- If $\Delta = 2k + 1$, then $|I_1| = |I_2| = |I_3| = k$
and $|I_4| = |I_5| = k + 1$.

Fig. 1. The Erdős-Nešetřil construction.

Although Conjecture 1.3 is maybe one of the most important questions related to strong edge coloring, it is still widely open. However, it is known to be true whenever $\Delta \leq 3$, as proved notably by Andersen [1]. For $\Delta = 4$, the best result towards Conjecture 1.3 is due to Cranston, who proved the following.

Theorem 1.4 (Cranston [4]) *Every graph G with $\Delta = 4$ satisfies $\chi'_s(G) \leq 22$.*

He also proved that one color could be saved when the graph is not 4-regular. Nonetheless, there is still a small gap between Theorem 1.4 and the upper bound suggested by Conjecture 1.3 for graphs with maximum degree 4, namely 20.

2 Our results

Toward Conjecture 1.3, we herein focus on the family of graphs with maximum degree 4. As mentioned above, the Erdős-Nešetřil graph with maximum degree 4 has strong chromatic index exactly 20. However, it is worth mentioning that, this extremal graph and its variations apart, it seems that we can save a significant number of colors. As an illustration of this statement, let us recall that the highest strong chromatic index found in a planar (resp. bipartite) graph with maximum degree 4 so far is 12 (resp. 16).

In this context, we consider graphs with maximum degree 4 and bounded *maximum average degree* (or *mad* for short), where the maximum average

degree of some graph G is defined as

$$\text{mad}(G) := \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \text{ is a subgraph of } G \right\}.$$

As a first step towards Conjecture 1.3 for $\Delta = 4$, we confirm it for graphs of mad less than $\frac{19}{5}$. We then provide a number of situations where sparser graphs can be colored with fewer colors. To summarize our results, we prove here the following.

Theorem 2.1 *For every graph G with $\Delta = 4$, we have:*

- (i) *if $\text{mad}(G) < \frac{16}{5}$, then $\chi'_s(G) \leq 16$;*
- (ii) *if $\text{mad}(G) < \frac{10}{3}$, then $\chi'_s(G) \leq 17$;*
- (iii) *if $\text{mad}(G) < \frac{17}{5}$, then $\chi'_s(G) \leq 18$;*
- (iv) *if $\text{mad}(G) < \frac{18}{5}$, then $\chi'_s(G) \leq 19$;*
- (v) *if $\text{mad}(G) < \frac{19}{5}$, then $\chi'_s(G) \leq 20$.*

Theorem 2.1 is proved through a discharging method using purely local arguments. The method consists in considering a minimum counter-example to the statement (duly reformulated so as to avoid some technicalities), proving that it satisfies some structural constraints (*e.g.* there is no vertex of degree 1, nor two adjacent vertices of degree 2, etc.), and then use a discharging argument to claim that a graph with those structural properties cannot satisfy the sparseness hypothesis.

3 Conclusions: sharpness and further work

We now discuss the tightness of the upper bounds in Theorem 2.1. For this purpose, we provide some examples of graphs with maximum degree 4 and different values of mad , and compare their strong chromatic index to the upper bounds of Theorem 2.1.

As mentioned earlier, the Erdős-Nešetřil graph with maximum degree 4 notwithstanding, it seems difficult to exhibit graphs with maximum degree 4 and strong chromatic index close to 20. Again, as already mentioned, such graphs should not be planar (unless there exist worst examples than those exhibited in [7]), nor bipartite (unless Conjecture 1.2 is wrong). This task seems even more complicated when a condition on the maximum average degree must be met: for a graph with maximum degree 4 to have strong chromatic index close to 20, a lot of 4-vertices seem necessary. As a consequence, the

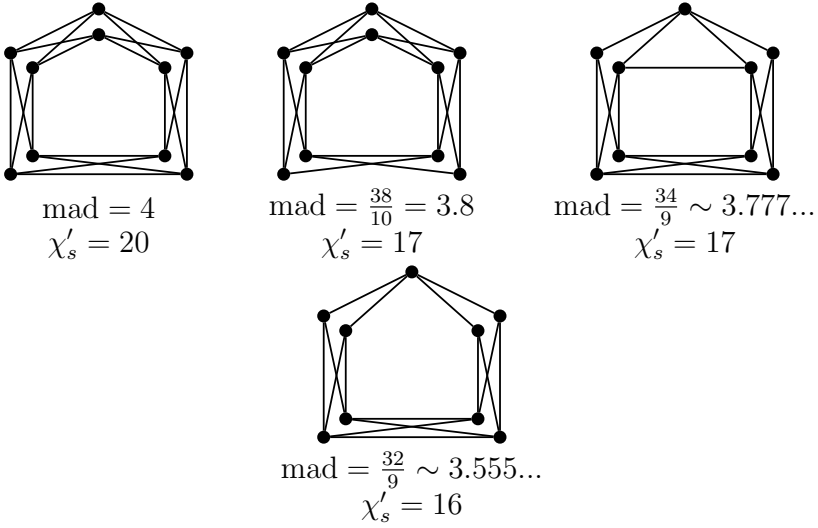


Fig. 2. Some variations of the Erdős-Nešetřil graph with $\Delta = 4$.

maximum average degree is naturally close to 4.

Some explicit graphs with maximum degree 4, maximum average degree strictly less than 4, and large strong chromatic index are depicted in Figure 2. These result from straightforward modifications of the Erdős-Nešetřil graph with $\Delta = 4$. These graphs give already an idea of how far from the optimal the upper bounds in Theorem 2.1 may be. In particular, if we compare the values given in Theorem 2.1 to the characteristics of these sample graphs, we come up with Table 1.

# of colours	proved for $\text{mad} <$	false for $\text{mad} \geq$
16	$\frac{16}{5} = 3.2$	$\frac{32}{9} \sim 3.555\dots$
17	$\frac{10}{3} \sim 3.333\dots$	$\frac{38}{10} = 3.8$
18	$\frac{17}{5} = 3.4$	4
19	$\frac{18}{5} = 3.6$	4
20	$\frac{19}{5} = 3.8$	4

Table 1

On the tightness of Theorem 2.1 in terms of mad and χ'_s .

Beside the obvious search for optimal bounds, it would be interesting to generalize the bounds of Theorem 2.1 to any value of Δ . In particular, we

believe that the following statement would be a reasonable first step toward Conjecture 1.3.

Conjecture 3.1 *Every graph G with no subgraph of average degree at least $\frac{\Delta(G)}{2}$ satisfies $\chi'_s(G) \leq \frac{5\Delta(G)^2}{4}$.*

References

- [1] L.D. Andersen. The strong chromatic index of a cubic graph is at most 10. *Discrete Mathematics*, 108(1-3):231-252, 1992.
- [2] J. Bensmail, A. Harutyunyan, H. Hocquard, P. Valicov. Strong edge-colouring of sparse planar graphs. *Discrete Applied Mathematics*, 179:229-234, 2014.
- [3] J. Bensmail, A. Lagoutte, P. Valicov. Strong edge-coloring of $(3, \Delta)$ -bipartite graphs. Preprint available online at <http://arxiv.org/abs/1412.2624>.
- [4] D.W. Cranston. Strong edge-coloring of graphs with maximum degree 4 using 22 colors. *Discrete Mathematics*, 306(21):2772-2778, 2006.
- [5] P. Erdős, J. Nešetřil. Irregularities of partitions (G. Halász, V.T. Sós, Eds.), [Problem], 162-163, 1989.
- [6] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza. Induced matchings in bipartite graphs. *Discrete Mathematics*, 78:83-87, 1989.
- [7] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza. The strong chromatic index of graphs. *Ars Combinatoria*, 29B:205-211, 1990.
- [8] J.L. Fouquet, J.L. Jolivet. Strong edge-colorings of graphs and applications to multi- k -gons. *Ars Combinatoria*, 16A:141-150, 1983.
- [9] H. Hocquard, P. Ochem, P. Valicov. Strong edge-colouring and induced matchings. *Information Processing Letters*, 113(19-21):836-843, 2013.
- [10] D.B. West. Strong edge-coloring. Open Problems - Graph Theory and Combinatorics, <http://www.math.uiuc.edu/~west/openp/strongedge.html>.
- [11] M. Mahdian. The strong chromatic index of graphs, M.Sc. Thesis, Department of Computer Science, University of Toronto, 2000.