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Drawing a disconnected graph on the torus (Extended abstract)

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Abstract

We study drawings of graphs on the torus with crossings allowed. A question posed in [4], specialized to the case of the torus, asks, whether for every disconnected graph there is a drawing in the torus with the minimal number of crossings, such that one of the graphs is drawn in a planar disc. We reduce the problem to an interesting question from the geometry of numbers and solve a special case.

Keywords: graph drawing, crossing number, torus, disconnected graph

1 Introduction

Planarity is a central topic in the graph theory. There are at least two frequently used ways to extend this notion: we may try to embed a graph without a crossing on a different surface than the plane and we may draw the graph in the plane with as few crossings as possible. We are going to study the

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common extension of these two approaches. Following Širáň [6] and Archdeacon et al. [1] we define $\mathrm{cr}_i(G)$, the *i-th orientable crossing number of G* as the minimal number of crossings, when we draw G in the two-dimensional compact manifold (surface) of genus i—the sphere with i handles added. As it is usual, we assume that no three edges go though the same vertex, no edge goes though a vertex, and edges are simple curves in the surface. When two edges cross several times, we count all of the crossings.

The authors of [1,6] deal mainly with the issue of characterizing the *crossing sequences*—sequences $(\mathrm{cr}_i(G))_{i\geq 0}$. They prove that every sequence of integers that is decreasing (until it hits 0) and convex is the crossing sequence of some graph. Later, DeVos et al. [4] prove that every sequence $(a, b, 0, 0, \dots)$ ($a > b$ are positive integers) is a crossing sequence. In the same paper [4] the following problem is posed.

Problem 1.1 *Let H be a disjoint union of graphs G and G', and let $\mathcal{S}$ be a (possibly nonorientable) surface. Is there an optimal drawing of H on $\mathcal{S}$, such that no edge of G crosses an edge of G'?*

If we have a graph which is a disjoint union of G and G', then we can always "use part of the surface for G and the other part for G'"—technically, we obtain surface of genus i by glueing surfaces of genera j and $i - j$ along a circle and draw G in the first one and G' in the second one. This leads to

$$\mathrm{cr}_i(G \,\dot{\cup}\, G') \leq \min_j \bigl(\mathrm{cr}_j(G) + \mathrm{cr}_{i-j}(G')\bigr) .$$

If the answer to Problem 1.1 is positive, then this inequality is in fact an equality.

The answer to Problem 1.1 is trivial when $\mathcal{S}$ is the plane. It is not hard to prove it for $\mathcal{S}$ being a projective plane [4]. Beaudou et al. [2] prove the result for the Klein bottle. Here, we deal with the first orientable case, the torus.

Our main tool is a result of de Graaf and Schrijver [3] that provides an algebraic characterization of the minimal number of crossings. To state the result, we must introduce a bit of terminology first.

The crossing number $\mathrm{cr}(C, D)$ of two closed curves C and D is simply the total number of intersections. If G is a graph with a fixed drawing then $\mathrm{cr}(G, D)$ is the number of intersections between G and D. The following variants require the notion of homotopy (continuous deformation of one closed curve into another. We define $\mathrm{mincr}(C, D)$ to be the minimum of $\mathrm{cr}(C', D')$ for some curve C' homotopic to C and another one, D', homotopic to D. Finally, $\mathrm{mincr}(G, D)$ is the minimum of $\mathrm{cr}(G, D')$ for a curve D' homotopic to D.

The following appears as Theorem 76.1 in Schrijver's excellent monograph [5].

Theorem 1.2 *Let $G = (V, E)$ be an Eulerian graph embedded in a compact surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that*

$$\mathrm{mincr}(G, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$$

for each closed curve D on S.

2 Drawing graphs on the torus

In this section we reduce Problem 1.1 for the torus to an inequality, although a rather unwieldy one. We need to introduce some notation first. For vectors $u, v \in \mathbb{R}^2$ we let (u, v) be the 2×2 matrix with columns u and v. Given two collections (i.e., multisets) of vectors, X and X', we define

$$L(X, X') = \sum_{u \in X} \sum_{v \in X'} |\det(u, v)|.$$

Further, we define

$$N(X) = \sum_{u \in X} |u_1| \cdot \sum_{v \in X} |v_2|,$$

where u_1, u_2 are the two coordinates of u. For any 2×2 matrix A we define $N_A(X) = N(AX)$, where AX is the collection $\{Av \mid v \in X\}$. Finally, we define $M(X)$ to be the minimal value of $N_A(X)$, when A is a 2×2 unimodular matrix (that is, an integer matrix of determinant ± 1). It is a simple exercise to show that $M(X)$ is, indeed, well-defined, whenever X is finite.

Problem 2.1 *Let X, X' be two finite collections of vectors in $\mathbb{Z}^2$. Is it true that*

$$L(X, X') \geq \min\{M(X), M(X')\}?$$

Theorem 2.2 *If the answer to Problem 2.1 is positive, then also the answer to Problem 1.1 is positive.*

Proof. Consider an optimal drawing of $H = G \cup G'$ on the torus; that is, one (of possibly many) drawings with the minimal number of crossings.

If two edges of G intersect, we may subdivide these two edges and identify the two new vertices. Repeating as necessary, and using the same procedure for G', we arrive to the case where G and G' are without self-intersections. It

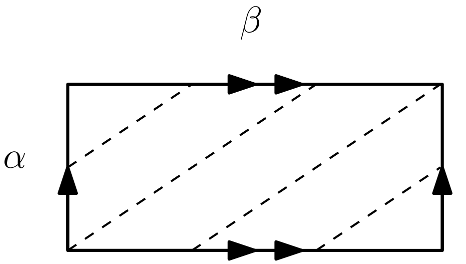


Fig. 1. Illustration of homotopy types of curves on the torus. The basis curves α and β together with a curve of type $3\alpha + 2\beta$.

follows that we may assume, that both G and G' are embedded in the torus. Our goal is to prove that one of G and G' may be embedded in a planar disc with at most the same number of crossings as our drawing of H has.

Next, we double each edge of G and of G'. We draw the new edges very closely to the original ones, so that the number of crossings is increased four-times. It is easy to prove that if the original drawing was optimal than we have obtained an optimal drawing of the "doubled" graph. Thus, we may assume G and G' are Eulerian graphs.

Now we are in position to apply Theorem 1.2, twice. Using it, we decompose the edges of G into closed curves $C_1, \dots, C_k$ and the edges of G' into curves $C'_1, \dots, C'_k$.

To proceed further, we need to classify the homotopy types of the curves. To this end, we fix two noncontractible curves in the torus, say α and β, in such a way that they have exactly one point in common. It is well-known that every closed curve in the torus is homotopic to $m\alpha + n\beta$ for some integers m, n (see Figure 1). We let (m_i, n_i) be the "coordinates" of the curve C_i and define (m'_i, n'_i) similarly. Note that we need to choose an (arbitrary) orientation of each of the curves, so the vectors (m_i, n_i), (m'_i, n'_i) are only defined up to multiplication by -1.

Let C and C' be curves of type (m, n) and (m', n'). It is easy to verify that $\operatorname{mincr}(C, C') = \left|\det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix}\right|$. Two applications of the formula in Theorem 1.2 yield that the number of crossings between G and G' is at least $L(X, X')$. Here we use the function L defined in the beginning of this section. We put $X = \{(m_i, n_i)^T : i = 1, \dots, k\}$ and define X' similarly.

Let us compare this to drawing one graph in the torus without any crossing at all (this is possible by our assumptions) and the other, say G, in a planar disc. There are many planar drawings of G, we will only use a simple type

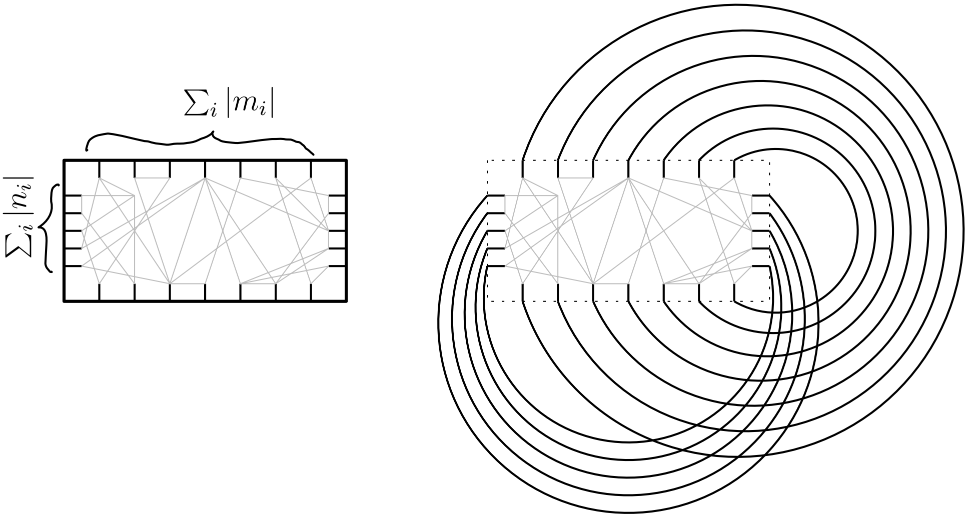


Fig. 2. Illustration of the planar way to draw a toroidal graph.

obtained from its toroidal drawing.

We represent the torus as a rectangle with a "wrap-around". From the notation we introduced for the curves C_i we see that the top and bottom side of the rectangle are intersected $(\sum_{i=1}^{k} |m_i|)$-times, while the left and the right one $(\sum_{i=1}^{k} |n_i|)$-times (we may assume that no curve goes through the corners).

Thus, when we connect these points by curves in a plane, as depicted in Figure 2, the number of intersections that we introduce is $N(X)$.

We may possibly do better, though. We do have a choice how to cut the torus to get a rectangle. To do so, we may choose a different pair of closed curves than α and β. We choose curves $x_1\alpha + y_1\beta$ and $x_2\alpha + y_2\beta$ such that x_i, y_i are integers and $\det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \pm 1$. In other words, $B = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ is an unimodular matrix. It is easy to see that to transform the "coordinates" X we need to multiply by $A = B^{-1}$, another unimodular matrix. Thus, we may draw G in a planar disc with $N_A(X)$ crossings for any unimodular matrix A. Also, we may choose to draw in a planar way G' instead of G. Consequently, if $L(X, X') \geq \min\{M(X), M(X')\}$, we certainly can achieve the same or better number of crossings with a drawing where G and G' do not intersect.

3 Reducing to a finite number of variables

In this section we reduce the inequality in Problem 2.1 to a simpler one, with only finitely many variables. The main idea of the reduction is that we collect together vectors of X that are in the same part of the plane. We will not reduce the general case though, we need to assume $X = X'$. Arguable, this is the most difficult case, as the quantity $L(X, X')$ seems smallest in such case. Unfortunately, we have been unable (yet) to transform this feeling into a solution of Problem 2.1 in the general case.

Lemma 3.1 *Define a cone P in $\mathbb{R}^8$:*

$$\begin{aligned} P = \{(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) \in \mathbb{R}^8 \ : \ & x_1 \geq 0, x_2 \geq 0, x_3 \leq 0, x_4 \leq 0, \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, \\ & x_1 \geq y_1, x_2 \leq y_2, x_3 + y_3 \geq 0, x_4 + y_4 \leq 0, \\ & y_1 + y_2 + y_3 - y_4 \geq -2x_3, \\ & x_1 + x_2 + x_3 - x_4 \geq 2y_4, \\ & -y_1 + y_2 + y_3 + y_4 \geq 2x_2, \\ & x_1 - x_2 - x_3 - x_4 \geq 2y_1\} \end{aligned}$$

Define a quadratic function $f(x_1, \dots, y_4)$ as

$$(x_1 - x_2 - x_3 - x_4)y_2 + (x_1 + x_2 + x_3 - x_4)y_3 + (x_1 + x_2 + 3x_3 + x_4)y_4 - (x_1 + 3x_2 + x_3 + x_4)y_1.$$

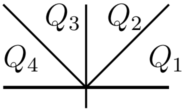
Then $f \geq 0$ on P.

First we show how to apply this lemma to prove a special case of our Problem 2.1. In the graph-drawing setting, this case corresponds to the two components, G and G' being isomorphic and in drawn in the same way.

Theorem 3.2 *The answer to Problem 2.1 is positive whenever $X = X'$.*

Proof. First, we assume that X is such that the $M(X) = N(X)$. That is, the minimal value of $N_A(X)$ is attained for $A = I$. Clearly, this is without loss of generality, as we may possibly replace X with AX.

Next, we note that neither $L(X, X)$ nor $M(X)$ change if we multiply some of the vectors by a -1. Thus, we may assume that for every $v \in X$ the second coordinate, v_2, is nonnegative. We partition the collection X as $Q_1 \dot\cup Q_2 \dot\cup Q_3 \dot\cup Q_4$ according to the sign of v_1, $v_1 + v_2$, and $v_1 - v_2$; see the figure below. (Vertices on the boundary are assigned arbitrarily.)



For $k = 1, \ldots, 4$ we put $(x_k, y_k)^T = \sum_{v \in Q_k} v$. The first eight inequalities in the definition of P are a simple consequence of this definition.

From the minimality of $N_I(X)$ it follows that for for each matrix $A \in \left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \right\}$ we have $N(AX) \geq N(A)$. When we express this in terms of the x_i's and y_i's, we get the remaining eight inequalities in the definition of P. It remains to see that Lemma 3.1 guarantees that $L(X, X) \geq N(X)$. First, we observe that

$$L(X,X) = \sum_{u \in X} \sum_{v \in X} |\det(u,v)| = \sum_{i,j=1}^{4} \sum_{u \in Q_i, v \in Q_j} |\det(u,v)| \geq$$
$$\geq \sum_{i,j=1}^{4} \Big| \sum_{u \in Q_i, v \in Q_j} \det(u,v) \Big| = \sum_{i,j=1}^{4} \left| \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \right|.$$

Similarly,

$$N(X) = \sum_{i,j=1}^{4} |x_i| \cdot |y_j| = (x_1 + x_2 - x_3 - x_4) \cdot (y_1 + y_2 + y_3 + y_4).$$

A bit of calculation reveals that f in the statement of Lemma 3.1 is the difference of the last terms of the previous two equations; thus $L(X, X) - N(X) \geq f(x_1, \ldots, y_4)$. It follows $L(X, X) \geq N(X)$, which finishes the proof.

4 Solving the inequality

In this section we prove Lemma 3.1. There are several general methodologies to solve this type of inequality: Groebner basis and Positivestellensatz being the top two. Unfortunately, we have been unable to make these techniques work for our case: at some point, we needed a computer assistance and the program ran out of memory after longish computation. Thus, we settle for a very down-to-earth approach that begs for an improvement. Due to homogeneity of the quadratic function f we only need to verify that $f \geq 0$ on $P \cap [-1, 1]^8$. To this end, we decompose this set into 2^{16} parts based on which of the defining inequalities are satisfied with an equality. Each of these subproblems is solved by an easy differentiation and the extremal point is checked for being nonnegative. The calculations were made in the computer algebra system Sage [7], the file needed to reproduce them is available at the third author's web page [8].

5 Conclusion

We approach the problem of drawing disconnected graphs on torus, Problem 1.1, by algebraic means: Problem 2.1. Then we solve a special case of Problem 2.1 by reducing to a finite number of variables and solving that by using a computer algebra system. The special case corresponds to the case when the graph we are drawing consists of two isomorphic components, say $G = G_1 \cup G_2$. Our result shows that in an optimal drawing of G, the drawings of G_1 and of G_2 cannot be isomorphic. More importantly, our result suggests, that approaching Problem 1.1 via Problem 2.1 is a viable option.

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