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Partitioning 3-edge-coloured complete bipartite graphs into monochromatic cycles

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Abstract

We show that any colouring with three colours of the edges of the complete bipartite graph $K_{n,n}$ contains 18 vertex-disjoint monochromatic cycles which together cover all vertices. The minimum number of cycles needed for such a covering is five, and we show that this lower bound is asymptotically true. This extends known results for complete graphs.

Keywords: monochromatic cycle partition, edge-coloured graph, complete graph, complete bipartite graph

1 Introduction

1.1 Cycle partitioning complete graphs

Given an arbitrary colouring of the edges of a graph G with r colours, we are interested in determining the smallest number of monochromatic cycles that partition the vertex set of G. (Here, an r-edge-colouring is not necessarily proper, and cycles may be single vertices, edges or the empty set.) Lately, this question received a considerable amount of attention from the community.

For r-edge-coloured complete graphs K_n, an easy construction shows that at least r cycles are necessary to cover all the vertices, and Erdős, Gyárfás and Pyber [5] show that, no matter how large the graph is, $cr^2 \log r$ cycles always suffice, where c is some absolute constant. The currently best known upper bound of $100r \log r$ monochromatic cycles is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [6]. In [5], it was conjectured that actually r cycles suffice, which, for $r = 2$, had been suggested earlier by Lehel [2] in a stronger form (the two cycles should have different colours). Lehel's conjecture was settled for large n in [14,1], and for all n in [4]. For $r = 3$, Gyárfás, Ruszinkó, Sárközy and Szemerédi [9] showed the following theorem.

Theorem 1.1 *For any 3-edge-colouring of K_n,*

(a) *there is a partition of all but $o(n)$ vertices of K_n into 3 monochromatic cycles, and*

(b) *if n is large enough, then the vertices of K_n can be partitioned into 17 monochromatic cycles.*

Actually, by a slight modification of their method, one can replace the number 17 with 10. However, the conjecture of Erdős, Gyárfás and Pyber was finally disproved by Pokrovskiy [16] for all $r \geq 3$. In his examples, there are partitions of all but exactly one vertex into r monochromatic cycles. In light of this, several authors [3,16] proposed toning down the conjecture, allowing for a constant number of uncovered vertices.

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[1]

[2] Supported by Fondecyt Regular no. 1140766.

1.2 Cycle partitioning complete bipartite graphs

For the balanced complete bipartite graph $K_{n,n}$ whose edges are coloured with r colours, $2r-1$ is the best known lower bound for the number of cycles needed to cover all the vertices. Indeed, take a properly r-edge-coloured $K_{r,r}$, replace each of the vertices in one partition class with r new vertices and replace one vertex in the other class with $(r(r-1)+1)$ new vertices. Let this new graph H be complete bipartite and colour the edges in such a way that each edge of H has the same colour as its projection onto $K_{r,r}$. Since no two of the unreplaced vertices lie on a common monochromatic cycle, we need $r-1$ cycles to cover all of them. As the vertices of the other side of $K_{n,n}$ have been blown up to the size of r each, we need r more cycles to cover the rest of the graph. Note that we can further blow up each vertex equally to give examples of graphs which can not be covered by 4 monochromatic cycles and additional $o(n)$ single vertices. A similar construction is given in [16]. For the upper bound for general r, Haxell [10], answering a question from [5], showed that $O((r\log r)^2)$ monochromatic cycles suffice to partition all the vertices. This bound has been improved to $O(r^2\log r)$ by Peng et al. [15] and recently in a more general setting to $O(r^2)$ [12].

For $r=2$, Pokrovskiy [16] showed that any 2-edge-coloured $K_{n,n}$ can be partitioned into 2 monochromatic *paths*, unless the colouring is a so-called split colouring. A split colouring of $K_{n,n}$ is a colouring of the edges such that there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a properly edge-coloured $K_{2,2}$. In the case of a split colouring, one quickly notes that three monochromatic cycles (or paths) are always enough to cover all vertices, and this is best possible. Further, in [17] in is shown that 12 monochromatic cycles suffice to partition all the vertices, and with three cycles we can partition all but $o(n)$ of the vertices. Haxell's [10] proof yields that in the case $r=3$, 1695 monochromatic cycles suffice to partition all vertices.

We improve this bound, and also show that asymptotically five cycles are enough, which is best possible by the above construction. Our main result is bipartite version of Theorem 1.1:

Theorem 1.2 (Main result) *For any 3-edge-colouring of $K_{n,n}$,*

(a) *there is a partition of all but $o(n)$ vertices of $K_{n,n}$ into five monochromatic cycles, and*

(b) *if n is large enough, then the vertices of $K_{n,n}$ can be partitioned into 18 monochromatic cycles.*

The remainder of this abstract is devoted to an outline of the proof of

Theorem 1.2.

2 Partitioning into 18 cycles

The proof of Theorem 1.2 follows roughly the strategy of the proof of Theorem 1.1. The main difference is that at some point the bipartite setting requires us to switch or extend some of the arguments. In the following two subsections we outline the proofs of Theorem 1.2(a) and (b). We finish with a sketch of the proof of our key lemma in Subsection 2.3.

2.1 Proof of Theorem 1.2(a)

The proof of Theorem 1.2(a) involves the construction of large monochromatic connected matchings and an application of the regularity lemma [18], the combination of which has been introduced by Łuczak [13] and has become a standard approach. A *monochromatic connected matching* is a matching in a connected component of the graph spanned by the edges of a single colour. (And such a component is called a *monochromatic component.*) The following is our key lemma, which takes up most of the proof – see Subsection 2.3.

Lemma 2.1 *Let the edges of $K_{n,n}$ be coloured with three colours. Then there is a partition of the vertices of $K_{n,n}$ into five or less monochromatic connected matchings.*

Now for the proof of Theorem 1.2(a), assume we are given a 3-edge-coloured $K_{n,n}$. We apply the regularity lemma, and the majority colours between the clusters induce a 3-edge-colouring of the reduced graph R which is almost complete bipartite. We then use a robust version of Lemma 2.1, which permits us to partition almost all of R into five monochromatic connected matchings. In the subsequent step, we apply a specific case of the blow-up lemma (see [11,13,8]) to get from our matchings in R to five monochromatic cycles in $K_{n,n}$ which together partition almost all of the vertices.

2.2 Proof of Theorem 1.2(b)

The proof of Theorem 1.2(b) combines ideas of Haxell [10] and Gyárfás et al. [9] with Theorem 1.2(a). First, we fix a large monochromatic subgraph H, which has the property that it is Hamiltonian and remains so even if some of the vertices are deleted from it. Then, using Theorem 1.2(a), we cover almost all vertices of $K_{n,n} - V(H)$ with five vertex-disjoint monochromatic cycles. The amount of still uncovered vertices is much smaller than the order of H, this allows us to apply a lemma from [6] in order to absorb these vertices using only a few more cycles, running through vertices of H. We finish by taking one more monochromatic cycle, which covers the remains of H. Here are some more details:

We call a balanced bipartite subgraph H of a $2n$-vertex graph ϵ-*Hamiltonian*, if any balanced bipartite subgraph of H with at least $2(1-\epsilon)n$ vertices is Hamiltonian. The next lemma is a combination of results from [10,15].

Lemma 2.2 *For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph on $2n \geq 2n_0$ vertices and of edge density at least γ has a $\gamma/4$-Hamiltonian subgraph of size at least $\gamma^{3024/\gamma}n/3$.*

The next lemma is due to Gyárfás et al. It allows us to absorb small vertex sets with few monochromatic cycles.

Lemma 2.3 (Gyárfás et al. [7]) *For any $r \geq 1$, there is a constant $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $m \leq \frac{n}{(8r)^{8(r+1)}}$, and for any r-colouring of $K_{n,m}$, there are $2r$ disjoint monochromatic cycles covering all m vertices on the smaller side.*

Now to prove Theorem 1.2(b), assume that $K_{n,n}$ is 3-edge-coloured. Let A and B be the two partition classes of the 3-edge-coloured $K_{n,n}$. We assume that $n \geq n_0$, where we specify n_0 later. Pick subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ of size $\lceil n/2 \rceil$ each. Say red is the majority colour of $[A_1, B_1]$, the subgraph of $K_{n,n}$ induced by the vertex set $A_1 \cup B_1$. Lemma 2.2 applied with $\gamma = 1/3$ yields a red 1/12-Hamiltonian subgraph $[A_2, B_2]$ of $[A_1, B_1]$ with

$$|A_2| = |B_2| \geq 3^{-9999}|A_1| \geq 3^{-10^4}n.$$

Set $H := G - (A_2 \cup B_2)$, and note that each vertex class of H has order at least $\lfloor n/2 \rfloor$. Let $\delta := 24^{-32} \cdot 3^{-10^4}$. Assuming n_0 is large enough, Theorem 1.2(a) yields five monochromatic vertex-disjoint cycles covering all but at most $2\delta n$ vertices of H. If there are any isolated vertices among these cycles, we extend them to edges for sakes of balancedness. Let $X_A \subseteq A$ (resp. $X_B \subseteq B$) be the set of uncovered vertices in A (resp. B). We then have $|X_A| = |X_B| \leq \delta n$.

By the choice of δ, and since we assume n_0 to be sufficiently large, we can apply Lemma 2.3 to the bipartite graphs $[A_2, X_B]$ and $[B_2, X_A]$. We obtain a union $\mathcal{C}$ of twelve vertex-disjoint monochromatic cycles that together cover $X_A \cup X_B$. As $|X_A| = |X_B| \leq \delta n \leq 3^{-10^4}/12$, we know that $[A_2, B_2] - V(\bigcup \mathcal{C})$ contains a red Hamiltonian cycle, because $[A_2, B_2]$ is 1/12-Hamiltonian. Thus, in total, we covered G with at most $5 + 12 + 1 = 18$ vertex-disjoint monochromatic cycles.

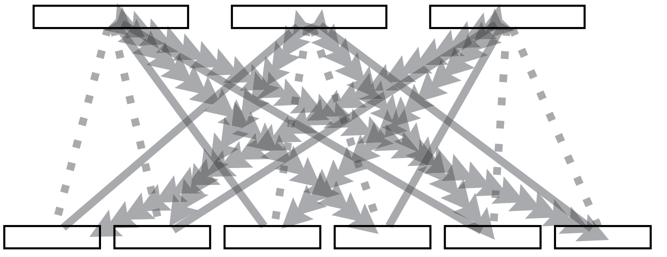

Fig. 1. The colouring of $K_{n,n}$. The edges represent complete bipartite graphs in the respective colour.

2.3 Proof of Lemma 2.1

The proof of Lemma 2.1 is involved, but purely combinatorial. Here we present some of the steps to give an idea of the argument. We remark that the proof of the robust version of Lemma 2.1 follows virtually the same pattern in a much more technical environment. Before we start, we need one more lemma, which covers the case of 2 colours. It follows from the above-mentioned result of [16].

Lemma 2.4 *Let the edges of $K_{n,n}$ be coloured in red and blue. Then $K_{n,n}$ can be covered with two vertex disjoint monochromatic connected matchings, one of each colour, or the colouring is split. In the latter case $K_{n,n}$ can be covered with one red and two blue, and also with two red and one blue vertex disjoint monochromatic connected matchings.*

Now let us assume that $K_{n,n}$ is edge-coloured in red, green and blue. We first reduce the colouring to a form, where we have more information about the monochromatic components. Our first step for instance is the following claim.

Claim 2.5 *Each colour has at least three non-trivial components.*

Proof. We assume the opposite for colour red, say. Let R_1 and R_2 be components in red (possibly identical and/or trivial) and assume that all other red components are trivial. Let M be a maximum red matching in $R_1 \cup R_2$. Then every edge in the balanced complete bipartite subgraph $K_{n,n} - V(M)$ is green or blue. Hence we can apply Lemma 2.4 to cover $K_{n,n} - V(M)$ with three vertex disjoint monochromatic connected matchings. □

After some intermediate steps, which involve similar arguments in a more elaborate form, we get that there is a colour, red say, that has exactly three non-trivial components. We then take a maximum red matching in $K_{n,n}$ and remove its vertices from the graph. The remaining graph is balanced and coloured in blue and green. If we can cover it with two disjoint monochromatic matchings, we are done. Therefore by Lemma 2.4 we can assume that its colouring is split. This gives us information about the colours of the edges inside the red components, which after some more analysis allows to conclude that each colour has exactly three components. By continuing this type of reasoning we can reduce the colouring to the form shown in Figure 1. It is then only a matter of estimating the sizes of the individual parts and some more case distinctions, to obtain that $K_{n,n}$ can be covered by 5 disjoint monochromatic connected matchings.

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