Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Decompositions of highly connected graphs into paths of any given length ⋆

F. Botler G. O. Mota M. T. I. Oshiro Y. Wakabayashi

Instituto de Matemática e Estatística
Universidade de São Paulo
São Paulo, Brazil

Abstract

We study the decomposition conjecture posed by Barát and Thomassen (2006), which states that, for each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(T)|$ divides $|E(G)|$, then G admits a partition of its edge set into classes each of which induces a copy of T. In a series of papers, starting in 2008, Thomassen has verified this conjecture for stars, some bistars, paths of length 3, and paths whose length is a power of 2. In 2014, we verified this conjecture for paths of length 5. In this paper we verify this conjecture for paths of any given length.

Keywords: graphs, edge-decomposition, highly connected, path decomposition

⋆ This research has been partially supported by Fapesp (Proc. 2013/03447-6), CNPq (Proc. 477203/2012-4 and 456792/2014-7), and MaCLinC Project of Numec/USP, Brazil. F. Botler is supported by Fapesp (Proc. 2011/08033-0 and 2014/01460-8), G. O. Mota is supported by Fapesp (Proc. 2013/11431-2 and 2013/20733-2), M. T. I. Oshiro is supported by Capes, and Y. Wakabayashi is partially supported by CNPq Grant (Proc. 303987/2010-3).
Email:{`fbotler|mota|oshiro|yw`}`@ime.usp.br`

1 Introduction

A set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of pairwise edge-disjoint subgraphs of a graph G is called a *decomposition* of G if these subgraphs cover the edge set of G. If H_i, for $1 \leq i \leq k$, is isomorphic to a graph H, then we say that $\mathcal{D}$ is an *H-decomposition* of G. When H is a path of length 2, it is easy to prove that a connected graph G admits an H-decomposition if and only if G has an even number of edges. On the other hand, Dor and Tarsi [6] proved that deciding whether a graph admits an H-decomposition is an NP-complete problem whenever H has a component with at least 3 edges. It is then natural to look for sufficient conditions for a graph G to admit an H-decomposition. When H is a tree, Barát and Thomassen [2] conjectured that high edge-connectivity (together with the obvious necessary condition on the number of edges) may suffice.

Conjecture 1.1 *For any fixed tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Barát and Thomassen [2] proved that Conjecture 1.1 in the special case T is the claw $K_{1,3}$ is equivalent to a weakening of Tutte's 3-flow conjecture, posed by Jaeger [7]. Recently, Lovász, Thomassen, Wu, and Zhang [9] proved that a $(3k-3)$-edge-connected graph G admits a $K_{1,k}$-decomposition if $|E(G)|$ is divisible by k, showing that Conjecture 1.1 holds for stars, and, in particular, confirming Jaeger's weak 3-flow conjecture. Thomassen [11,12,13,14] also proved that Conjecture 1.1 holds for paths of length 3, paths of length 4, a family of bistars, and more recently, for paths whose length is a power of 2. The authors [4] and, independently, Merker [10] proved that Conjecture 1.1 holds for paths of length 5. In [10], Merker also verified the conjecture for trees with diameter at most 4.

A recent result, obtained by Barát and Gerbner [1] and independently by Thomassen [13], states that it is sufficient to prove Conjecture 1.1 for bipartite graphs. We consider the following slightly different (but also equivalent) version of Conjecture 1.1.

Conjecture 1.2 *For any fixed tree T, there exists a natural number k'_T such that, if G is a k'_T-edge-connected bipartite graph and $|E(G)|$ is divisible by $2|E(T)|$, then G admits a T-decomposition.*

In this paper we verify Conjecture 1.2 for paths of any given length. Our proof uses a generalization of the technique we presented in [4], which combines a method introduced by Thomassen [11] and a technique used by Lovász [8] for decomposition into cycles and paths.

In Section 2 we prove that a bipartite highly edge-connected graph admits a factorization that has some important properties. In Section 3 we present an important tool that allows us to switch edges among some trails of a fixed length and obtain paths of the same length. In Section 4 we show how to use these factorizations to obtain the desired decomposition. Owing to space limitation, we present only sketches of the proofs.

The basic terminology and notation used in this paper are standard (see, e.g. [5]). A *path* P in G is a sequence of distinct vertices $P = v_0v_1 \cdots v_k$ such that $v_iv_{i+1} \in E(G)$, for $i = 0, 1, \ldots, k-1$. The *length* of P is the number of its edges. A path of length k is denote by P_k. When convenient, we may refer to a path $P = v_0v_1 \cdots v_k$ as the subgraph of G induced by the edges v_iv_{i+1} for $i = 0, \ldots, k-1$. A *vanilla trail* is a trail $v_0v_1 \cdots v_k$ such that $v_1 \cdots v_{k-1}$ is a path. A *vanilla k-trail* is a vanilla trail of length k.

2 Fractional Factorizations and Bifactorizations

In this section we define the notions of factors and factorizations to deal with the special bipartite graphs considered here.

Definition 2.1 Let G be a graph and consider $X \subseteq V(G)$. Let a, k be positive integers such that $a \leq k$. We say that a subgraph F of G is an (X, a, k)-factor of G if $d_F(v) = (a/k)d_G(v)$ for every vertex v in X. We say that a decomposition $\mathcal{F}$ of G is an (X, k)-fractional factorization if $\mathcal{F}$ contains exactly two elements that are $(X, 1, k)$-factors of G, and $k/2-1$ elements that are Eulerian $(X, 2, k)$-factors of G.

Note that a necessary condition for a graph G to contain an $(X, 1, k)$-factor is that the degree in G of each vertex of X be divisible by k. Moreover, k must be even for G to admit an (X, k)-fractional factorization.

In what follows, we show how to obtain such factorizations for bipartite graphs in which for a given subset X of its vertices, each vertex in X has degree divisible by a fixed positive integer. The next lemma, which can be proved similarly to Proposition 2 in [13], shows how to obtain these graphs from highly edge-connected bipartite graphs.

Lemma 2.2 *Let $k \geq 2$ and r be positive integers. If $G = (A_1, A_2; E)$ is a $(6k-6+4r)$-edge-connected bipartite graph and $|E|$ is divisible by k, then G admits a decomposition into two spanning r-edge-connected graphs G_1 and G_2*

such that, in each graph G_i the degree of every vertex of A_i is divisible by k.

The next lemma presents a sufficient condition in terms of edge-connectivity for the graph G_i to admit an (A_i, k)-fractional factorization, for $i = 1, 2$.

Lemma 2.3 *Let k be an even positive integer and let $r = 16\max\{8, k+2\}$. Let $G = (A_1, A_2; E)$ be a bipartite graph such that the degree of each vertex in A_1 is divisible by k. If G is r-edge-connected, then G admits an (A_1, k)-fractional factorization.*

The symmetric properties of G_1 and G_2 lead naturally to the concept of bifactorization, that will be useful in this context.

Definition 2.4 Let k be an even positive integer and let $G = (A_1, A_2; E)$ be a bipartite graph. We say that a pair $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ of two sets of subgraphs of G is a k-bifactorization of G if $\mathcal{F}_1 \cup \mathcal{F}_2$ is a decomposition of G and, for $i \in \{1, 2\}$, $\mathcal{F}_i$ is an (A_i, k)-fractional factorization of $\cup_{F \in \mathcal{F}_i} F$.

Suppose $G = (A_1, A_2; E)$ is a bipartite graph such that $|E|$ is divisible by k. A direct implication of Lemmas 2.2 and 2.3 is that if G is $(6k - 6 + 4r)$-edge-connected for $r \geq 16\max\{8, k+2\}$, then G admits a k-bifactorization.

3 The Disentangling Lemma

In this section we present an important tool for the proof of our main result. For that, we introduce some new concepts and notation. Let $\mathcal{D}$ be a decomposition of a graph G. Let v be a vertex of G and let $vu \in E(G)$ and $T \in \mathcal{D}$ be such that $vu \in E(T)$. If $d_T(u) = 1$, we say that vu is a *hanging edge* of $\mathcal{D}$ at v. We denote by $\text{Hang}(v, \mathcal{D})$ the number of hanging edges of $\mathcal{D}$ at v. We say that $\mathcal{D}$ is *k-complete* if $\text{Hang}(v, \mathcal{D}) > k$ for every v in $V(G)$. We use the next lemma to prove Lemma 3.2.

Lemma 3.1 *Let G be a graph, and $\mathcal{D}$ a decomposition of G into trails of length k. Let T be a trail in $\mathcal{D}$, and v a vertex of T. If $\text{Hang}(v, \mathcal{D}) > k$, then there is a hanging edge vu of $\mathcal{D}$ at v such that $u \notin V(T)$.*

Let $\mathcal{D}$ be a decomposition of a graph G into trails. Given a vertex v of G, we denote by $\mathcal{D}(v)$ the number of elements of $\mathcal{D}$ that have v as an end-vertex. If an element T of $\mathcal{D}$ is such that v is the only end-vertex of T, we count T twice (or with multiplicity 2) in $\mathcal{D}(v)$.

Now we are ready to state the Disentangling Lemma, and sketch a proof for it.

Lemma 3.2 (Disentangling Lemma) *Let G be a bipartite graph. If G admits a k-complete decomposition $\mathcal{D}$ into vanilla k-trails, then G admits a k-complete decomposition $\mathcal{D}'$ into paths of length k such that $\mathcal{D}'(v) = \mathcal{D}(v)$ for every v in $V(G)$.*

Sketch of the proof. For every vanilla trail T of G, let $\tau(T)$ be the number of end-vertices of T that have degree greater than 1 in T. Let $\mathcal{D}$ be a k-complete decomposition into vanilla k-trails that minimizes $\sum_{T\in\mathcal{D}} \tau(T)$. Suppose that there is a vanilla trail T_0 in $\mathcal{D}$ that is not a path. Let x be an end-vertex of T_0 of degree greater than 1 in T_0, and let C be a cycle in T_0 that contains x. Let v be a neighbour of x in C. By Lemma 3.1, there is a hanging edge vu of $\mathcal{D}$ at v such that $u \notin V(T_0)$. Let T_1 be the element of $\mathcal{D}$ that contains vu. Now, let $T_0' = T_0 - vx + vu$, $T_1' = T_1 - vu + vx$, and $\mathcal{D}' = \mathcal{D} - T_0 - T_1 + T_0' + T_1'$. Note that $\mathcal{D}'(u) = \mathcal{D}(u)$ for every u in $V(G)$, and $\tau(T_0') = \tau(T_0) - 1$. If $\tau(T_1') \leq \tau(T_1)$, then $\mathcal{D}'$ is a k-complete decomposition of G into vanilla k-trails such that $\sum_{T\in\mathcal{D}'} \tau(T) < \sum_{T\in\mathcal{D}} \tau(T)$. Otherwise, we have that $\tau(T_1') = \tau(T_1) + 1$ and T_1' contains a cycle C' that contains xv. Let v' be a neighbour of x in C' such that $v' \neq v$. Now, repeat the above operation, as long as necessary, considering T_1' and v' instead of T_0 and v. We can show that this procedure halts, and we obtain the desired decomposition, concluding the proof.

4 Main result

In this section we show how to use bifactorizations of a graph to obtain a k-complete decomposition into vanilla k-trails. Once we have this, we may use Lemma 3.2 to obtain a P_k-decomposition. For that, we need one more definition. Let $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a $2k$-bifactorization of G, where $\mathcal{F}_1 = \{M_1, N_1, F_1, \ldots F_{k-1}\}$ and $\mathcal{F}_2 = \{M_2, N_2, H_1, \ldots H_{k-1}\}$. Let $G_i = \cup_{F\in\mathcal{F}_i} F$, for $i = 1, 2$. We say that $\mathcal{D}$ is $\mathbb{F}$-*balanced* if $\mathcal{D}(v) = d_{G_1}(v)/k + d_{M_2}(v) + d_{N_2}(v)$ for every v in A_1, and $\mathcal{D}(v) = d_{G_2}(v)/k + d_{M_1}(v) + d_{N_1}(v)$ for every v in A_2.

Proposition 4.1 *If $G = (A_1, A_2; E)$ is a bipartite graph that admits a 4-bifactorization $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$, then G admits an $\mathbb{F}$-balanced decomposition $\mathcal{D}$ into paths of length 2. Moreover, if the degree of every vertex in A_i is at least 8 in $G_i = \cup_{F\in\mathcal{F}_i} F$, for $i = 1, 2$, then $\mathcal{D}$ is 3-complete.*

Proof Let $\mathcal{F}_1 = \{M_1, N_1, F_1\}$ and $\mathcal{F}_2 = \{M_2, N_2, F_2\}$. Since $d_{M_1}(v) = d_{N_1}(v)$ for every vertex of A_1, we can decompose $M_1 \cup N_1$ into paths of length 2 with end-vertices in A_2. Since F_1 is Eulerian, we can decompose F_1 into paths of length 2 with end-vertices in A_1. We can find an analogous decomposition for

$M_2 \cup N_2$ and F_2. If $d_{G_i}(v) \geq 8$, then $d_{M_i \cup N_i}(v) \geq 4$ and $\mathrm{Hang}(v, \mathcal{D}) \geq 4$. This concludes the proof. □

Proposition 4.2 *If $G = (A_1, A_2; E)$ is a bipartite graph that admits a 6-bifactorization $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$, then G admits an $\mathbb{F}$-balanced decomposition $\mathcal{D}$ into paths of length 3. Moreover, if the degree of every vertex in A_i is at least 12 in $G_i = \cup_{F \in \mathcal{F}_i} F$, for $i = 1, 2$, then $\mathcal{D}$ is 3-complete.*

Proof Let $\mathcal{F}_1 = \{M_1, N_1, F_1, H_1\}$ and $\mathcal{F}_2 = \{M_2, N_2, F_2, H_2\}$. Choose an Eulerian orientation for F_1 and let $\mathcal{D}_{F_1}$ be a decomposition of F_1 into directed paths of length 2 with both end-vertices in A_1. Analogously, we obtain $\mathcal{D}_{H_1}$ from H_1. Since $d_{M_1}(v) = d_{N_1}(v) = d^+_{F_1}(v) = d^+_{H_1}(v)$ for every vertex of A_1, we can extend each path of $\mathcal{D}_{F_1}$ with an edge of M_1, and each path of $\mathcal{D}_{H_1}$ with an edge of N_1. Let $\mathcal{D}_1$ be the decomposition of G_1 obtained above. Note that the first and the last edge of every path of $\mathcal{D}_1$ is an edge of $M_1 \cup N_1$ and an edge of $F_1 \cup H_1$ that is oriented from A_2 to A_1, respectively. We can find an analogous decomposition $\mathcal{D}_2$ of G_2, and obtain the decomposition $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ of G, as desired. If $d_{G_i}(v) \geq 12$, then $d_{M_i \cup N_i}(v) \geq 4$ and $\mathrm{Hang}(v, \mathcal{D}) \geq 4$. This concludes the proof. □

Theorem 4.3 *Let k be a positive integer. Suppose $G = (A_1, A_2; E)$ is a bipartite graph that admits a $2k$-bifactorization $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$. If the degree of every vertex in A_i is at least $k(2k-1)$ in $G_i = \cup_{F \in \mathcal{F}_i} F$, for $i = 1, 2$, then G admits an $\mathbb{F}$-balanced k-complete P_k-decomposition.*

Sketch of the proof. Suppose the statement is not true and let k be the smallest positive integer for which the statement is false. Suppose that a graph G admits a 4-bifactorization $\mathbb{F}$ such that in G_i the vertices of A_i have minimum degree 6. Since every vertex of A_i must have degree multiple of 4 in G_i, we have that $d_{G_i}(v) \geq 8$ for every vertex $v \in A_i$. Thus, by Propositions 4.1, we have that $k > 2$. By Proposition 4.2, we conclude that $k > 3$. Let G be a bipartite graph and let $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{F}_1 = \{M_1, N_1, F_1, \ldots F_{k-1}\}$ and $\mathcal{F}_2 = \{M_2, N_2, H_1, \ldots H_{k-1}\}$. For $j = 1, 2$, choose an Eulerian orientation of F_{k-j}, let F^+_{k-j} be the edges of F_{k-j} oriented from A_2 to A_1, and let $F^-_{k-j} = F_{k-j} - F^+_{k-j}$. Let $\mathcal{F}'_1 = \{F^+_{k-2}, F^+_{k-1}, F_1, \ldots, F_{(k-2)-1}\}$ and $G'_1 = \cup_{F' \in \mathcal{F}'_1} F'$. Analogously, from $\mathcal{F}_2$ we can find $\mathcal{F}'_2$ and G'_2; then, we have that $\mathbb{F}' = (\mathcal{F}'_1, \mathcal{F}'_2)$ is a $2(k-2)$-bifactorization of $G' = G'_1 \cup G'_2$. Note that $d_{G'_i}(v) = \frac{k-2}{k} d_{G_i}(v) \geq (k-2)(2k-1) > (k-2)(2k-5)$. By the minimallity of k, we have that G' admits an $\mathbb{F}'$-balanced $(k-2)$-complete decomposition $\mathcal{D}'$ into paths of length $k-2$. Since $\mathcal{D}'$ is $\mathbb{F}'$-balanced, we have that $\mathcal{D}'(v) = d_{G'_1}/(k-2) + d_{H^+_{k-2}}(v) + d_{H^+_{k-1}}(v)$, for each vertex v of A_1. Since $d_{G'_1}(v)/(k-2) = d_{G_1}(v)/k =$

$d_{M_1}(v)+d_{N_1}(v)$ and $d_{H^+_{k-j}}(v) = d_{H^-_{k-j}}(v)$ for $j = 1,2$, we have $\mathcal{D}'(v) = d_{M_1}(v)+d_{N_1}(v) + d_{H^-_{k-2}}(v) + d_{H^-_{k-1}}(v)$, for every v in A_i. Analogously, we have $\mathcal{D}'(v) = d_{M_2}(v) + d_{N_2}(v) + d_{F^-_{k-2}}(v) + d_{F^-_{k-1}}(v)$ for every vertex v in A_2. Thus, using the edges in $M_1 \cup N_1 \cup H^-_{k-2} \cup H^-_{k-1}$ and $M_2 \cup N_2 \cup F^-_{k-2} \cup F^-_{k-1}$ we can extend each path of $\mathcal{D}'$ with an edge at each of its end-vertices, obtaining a decomposition $\mathcal{D}^*$ into vanilla k-trails. Since each path of $\mathcal{D}'$ receive an edge of $M_1 \cup N_1 \cup H^-_{k-2} \cup H^-_{k-1}$ or $M_2 \cup N_2 \cup F^-_{k-2} \cup F^-_{k-1}$, we have that $\mathcal{D}^*$ is $\mathbb{F}$-balanced. Choose $\mathcal{D}^*$ that maximizes $\sum_{v\in V(G)} \mathrm{Hang}(v,\mathcal{D}^*)$. Since $\mathcal{D}'(v) \geq d_{G_i}(v)/k \geq 2k-1$, one can prove that $\mathrm{Hang}(v,\mathcal{D}^*) > k$ for every vertex v in $V(G)$. Thus $\mathcal{D}^*$ is k-complete. By Lemma 3.2, there is a decomposition $\mathcal{D}$ of G into paths of length k such that $\mathcal{D}(v) = \mathcal{D}^*(v)$ for every vertex v in $V(G)$. This concludes the proof.

The next corollary, which is the main result of this paper, combines Lemmas 2.2 and 2.3, and Theorem 4.3.

Corollary 4.4 *Let $k \geq 2$ and $r = \max\{128, 32k+32, k(2k-1)\}$. Let $G = (A_1, A_2; E)$ be a bipartite graph such that $|E|$ is divisible by $2k$. If G is $(12k-6+4r)$-edge-connected, then G admits a decomposition into paths of length k.*

Proof By Lemma 2.2, G can be decomposed into two r-edge-connected graphs G_1 and G_2, such that $d_{G_i}(v)$ is divisible by $2k$, for every vertex v in A_i. Since $r \geq 16\max\{8, 2k+2\}$, by Lemma 2.3, G_i admits an $(A_i, 2k)$-fractional factorization $\mathcal{F}_i$. Thus, $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a $2k$-bifactorization for G. Since G_i is r-edge-connected, we have $d_{G_i}(v) \geq k(2k-1)$ for every vertex v in A_i. By Theorem 4.3, G admits an $\mathbb{F}$-balanced k-complete decomposition into paths of length k. □

5 Concluding remarks

The Disentangling Lemma (Lemma 2.3) can be used to obtain other decomposition results. In fact, in a forthcoming paper [3], we use a version of the Disentangling Lemma to prove results on path decompositions of regular graphs with prescribed girth. It would be very interesting to obtain a generalization of the Disentangling Lemma that deals with trees with maximum degree at most 3. Another interesting approach would be to obtain an extension of the results presented in Section 4 in order to obtain a decomposition of a highly edge-connected graph into graphs that can be obtained from a fixed tree by the identification of some of its vertices.

References

[1] Barát, J. and D. Gerbner, *Edge-decompositions of graphs into copies of a tree with four edges*, Electron. J. Combin. **21** (2014), p. Research Paper 55 pp. (electronic).

[2] Barát, J. and C. Thomassen, *Claw-decompositions and Tutte-orientations*, J. Graph Theory **52** (2006), pp. 135–146.

[3] Botler, F., G. O. Mota, M. T. I. Oshiro and Y. Wakabayashi, *Path decompositions of regular graphs with prescribed girth*, in preparation.

[4] Botler, F., G. O. Mota, M. T. I. Oshiro and Y. Wakabayashi, *Decomposition of highly connected graphs into paths of length 5*, ArXiv: 1505.04309 (2015).

[5] Diestel, R., "Graph theory," Graduate Texts in Mathematics **173**, Springer, Heidelberg, 2010, fourth edition, xviii+437 pp.

[6] Dor, D. and M. Tarsi, *Graph decomposition is NP-complete: a complete proof of Holyer's conjecture*, SIAM J. Comput. **26** (1997), pp. 1166–1187.

[7] Jaeger, F., *Nowhere-zero flow problems*, in: *Selected topics in graph theory, 3*, Academic Press, San Diego, CA, 1988 pp. 71–95.

[8] Lovász, L., *On covering of graphs*, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, New York, 1968 pp. 231–236.

[9] Lovász, L. M., C. Thomassen, Y. Wu and C.-Q. Zhang, *Nowhere-zero 3-flows and modulo k-orientations*, J. Combin. Theory Ser. B **103** (2013), pp. 587–598.

[10] Merker, M., "Decomposition of graphs," Ph.D. thesis, Technical University of Denmark, in preparation.

[11] Thomassen, C., *Decompositions of highly connected graphs into paths of length 3*, J. Graph Theory **58** (2008), pp. 286–292.

[12] Thomassen, C., *Edge-decompositions of highly connected graphs into paths*, Abh. Math. Semin. Univ. Hambg. **78** (2008), pp. 17–26.

[13] Thomassen, C., *Decomposing a graph into bistars*, J. Combin. Theory Ser. B **103** (2013), pp. 504–508.

[14] Thomassen, C., *Decomposing graphs into paths of fixed length*, Combinatorica **33** (2013), pp. 97–123.