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Packing, Counting and Covering Hamilton cycles in random directed graphs

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Abstract

A Hamilton cycle in a digraph is a cycle passing through all the vertices, where all the arcs are oriented in the same direction. The problem of finding Hamilton cycles in directed graphs is well studied and is known to be hard. One of the main reasons for this, is that there is no general tool for finding Hamilton cycles in directed graphs comparable to the so called Posá 'rotation-extension' technique for the undirected analogue. Here, we present a general and a very simple method, using known results, to attack problems of packing, counting and covering Hamilton cycles in random directed graphs, for every edge-probability $p > \log^C(n)/n$. Our results are asymptotically optimal with respect to all parameters and apply equally well to the undirected case.

Keywords: Hamilton cycles, Random graphs, Directed graphs.

1 Introduction

A *Hamilton cycle* in a graph is a cycle passing through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Once Hamiltonicity is established, there are many natural question to strengthen it. For example, one can ask the following questions:

- How many distinct Hamilton cycles does a given graph have? (This problem is referred to as the *counting* problem.)
- Let G be a graph with minimum degree $\delta(G)$. Is it possible to find roughly $\delta(G)/2$ edge-disjoint Hamilton cycles? (This problem is referred to as the *packing* problem.)
- Let $\Delta(G)$ denote the maximum degree of G. Is it possible to find roughly $\Delta(G)/2$ Hamilton cycles for which every edge $e \in E(G)$ appears in at least one of these cycle? (This problem is referred to as the *covering* problem.)

All the above mentioned questions have a long history and many results are known.

Let us define $\mathcal{G}(n,p)$ to be the probability space of graphs on a vertex set $[n] := \{1,\dots,n\}$, such that each possible (unordered) pair xy of elements of $[n]$ is appears as an *edge* independently with probability p. We say that a graph $G \sim \mathcal{G}(n,p)$ satisfies a property $\mathcal{P}$ of graphs with high probability (w.h.p.) if the probability that G satisfies $\mathcal{P}$ tends to 1 as n tends to infinity.

The question of packing in the probabilistic setting was firstly discussed by Bollobás and Frieze in the 80's [3]. They showed that if $\{G_i\}_{i=0}^{\binom{n}{2}}$ is a random graph process on $[n]$, where G_0 is the empty graph and G_i is obtained from G_{i-1} by adjoining a non-edge of G_{i-1} uniformly at random, w.h.p. as soon as G_i has minimum degree k (where k is a fixed integer), it has $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles plus a disjoint perfect matching if k is odd. Their result generalizes an earlier result of Bollobás [2] who proved (among other things) that for $p = \frac{\ln n + \ln\ln n + \omega(1)}{n}$, a typical graph $G \sim \mathcal{G}(n,p)$ is Hamiltonian. Note that this value of p is optimal in the sense that for $p =$

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$\frac{\ln n + \ln\ln n - \omega(1)}{n}$, it is known that w.h.p. a graph $G \sim \mathcal{G}(n,p)$ satisfies $\delta(G) \leq 1$, and therefore is not Hamiltonian. As the culmination of a long line of research Knox, Kühn and Osthus [15], Krivelevich and Samotij [17] and Kühn and Osthus [19] completely solved the packing problem for the entire range of p.

Regarding the question of counting, we mention the result of Cuckler and Kahn [5] who strengthened the classical theorem of Dirac from the 50's [6] and proved that every graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains at least $\left(\frac{\delta(G)}{e}\right)^n (1-o(1))^n$ distinct Hamilton cycles. A typical random graph $G \sim \mathcal{G}(n,p)$ with $p > 1/2$ shows that this estimate is sharp (up to the $(1-o(1))^n$ factor). Indeed, in this case with high probability $\delta(G) = pn + o(n)$ and the expected number of Hamilton cycles is $p^n(n-1)! < (pn/e)^n$.

These results are only few examples of a long line of research related to the Hamiltonicity property of random and pseudo-random graphs (see, e.g. [1], [9], [23], [16], [11], [12], [13]).

While in (general/random/pseudorandom) graphs there are many known results, much less is known about the problems of counting, packing and covering in the directed setting. The main difficulty here is that in this setting, the so called Posá rotation-extension technique (see, e.g. [22]) does not work in its simplest form. In the paper we present a simple method to attack (and to approximately solve) all the above mentioned problems in random/pseudorandom directed graphs, with an optimal (up to a $polylog(n)$ factor) density. Our method is also applicable in the undirected setting, and therefore reproves many of the known results in a simpler way.

A directed graph (or digraph) is a pair $D = (V, E)$ with a set of vertices V and a set of *arcs* E, where each arc is an ordered pair of elements of V. A Hamilton cycle in a digraph is a cycle going through all the vertices exactly once, where all the arcs are oriented in the same direction in a cyclic order. A directed graph is called *oriented*, if for every pair of vertices $u, v \in V$, at most **one** of the directed edges (u, v) or (v, u) appears in the graph.

Now, let us define $\mathcal{D}(n,p)$ to be the probability space consisting of all directed graphs on vertex set $[n]$ for which each possible *arc* is added with probability p independently at random.

The problem of packing Hamilton cycles in digraphs goes back to the 70's. Tilson showed that every complete digraph has a Hamilton decomposition. We also mention a remarkable result of Kühn and Osthus (see [18]) which proved that for any regular orientation of a sufficiently dense graph one can find a Hamilton decomposition. For many other results related to Hamiltonicity in digraphs see [21], [2], [10], [14], [7], [24], [25], [8], [4]. Our first result proves the existence of $(1-o(1))np$ edge-disjoint Hamilton cycles in $\mathcal{D}(n,p)$.

Theorem 1.1 *For $p=\omega\left(\frac{\log^4 n}{n}\right)$, w.h.p. the digraph $\mathcal{D}\sim\mathcal{D}(n,p)$ has $(1-o(1))np$ edge-disjoint Hamilton cycles.*

Note that this result is approximately tight (up to a polylog$n$ factor). At a cost of some polylog$n$ factor in the density, we obtain an analog for pseudo-random digraphs.

Theorem 1.2 *Let $\mathcal{D}$ be a (n,λ,p) pseudo-random digraph where $p=\omega\left(\frac{\log^{14} n}{n}\right)$. Then $\mathcal{D}$ contains $(1-o_\lambda(1))np$ edge-disjoint Hamilton cycles where $o_\lambda(1)\to 0$ as $\lambda\to 0$.*

We also show that in random directed graphs one can cover all the edges by not too many cycles.

Theorem 1.3 *Let $p=\omega\left(\frac{\log^4 n}{n}\right)$. Then, a digraph $\mathcal{D}\sim\mathcal{D}(n,p)$ w.h.p. can be covered with $(1+o(1))np$ directed Hamilton cycles.*

In the next theorem we show that the number of directed Hamilton cycles in such graphs (pseudorandom/random) is concentrated (up to a sub-exponential factor) around its mean.

Theorem 1.4 *Let $p=\omega\left(\frac{\log^2 n}{n}\right)$. Then, a digraph $\mathcal{D}\sim\mathcal{D}(n,p)$ w.h.p. contains $(1\pm o(1))^n n!p^n$ directed Hamilton cycles.*

We note that the method we used in Theorem 1.2 in order to extend the random result from Theorem 1.1 to the pseudo-random case can be also used in Theorem 1.3 and in Theorem 1.4 to get the pseudo-random versions of these theorems.

The proofs of the four theorems in the paper are based on very similar ideas and they all using the same method. Therefore, we will give here only one of the proof sketches. In the next section one can find a proof highlights for packing Hamilton cycles in $\mathcal{D}(n,p)$.

2 Proof sketch - Packing Hamilton cycles in $\mathcal{D}(n,p)$

Proof. [Proof of Theorem 1.1.] Let $\alpha = \alpha(n)$ be some function tending arbitrarily slowly to infinity with n. Let $n = m\ell + s$ where $\ell = \alpha \log n$, $t = \alpha^3 \log^3 n$ and let $p = \alpha^4 \log^4 n/n$. Let $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(t)}$ be a collection of partitions of $X = [n]$, chosen uniformly and independently at random, where $\mathcal{V}^{(i)} = S^{(i)} \cup V_1^{(i)} \cup \dots \cup V_\ell^{(i)}$, $|S^{(i)}| = s = n/\alpha \log n$ and $|V_j^{(i)}| = m$. To begin, whenever we expose the edges of a directed graph $\mathcal{D} \sim D(n,p)$, we will assign the edges of $\mathcal{D}$ among t edge disjoint subdigraphs $\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(t)}$. The digraphs $\mathcal{D}^{(i)}$ are constructed in a way that every directed edge between $V_j^{(i)}$ and $V_{j+1}^{(i)}$ in $\mathcal{D}^{(i)}$ appears with probability $(1-o(1))p\ell/t := p_{in}$ and all the directed edges $\overrightarrow{uv}$ ($u \in V_\ell^{(i)}$ and $v \in V_1^{(i)} \cup S^{(i)}$ or $u \in S^{(i)}$ and $v \in V_1^{(i)}$) appears independently with probability $(1-o(1))pn^2/\alpha t s^2$ (and we delete all other edges). We will prove that with probability $1-o(1/t)$, $\mathcal{D}^{(i)}$ contains $(1-o(1))np/t$ edge-disjoint Hamilton cycles.

Since every directed edge between $V_j^{(i)}$ and $V_{j+1}^{(i)}$ appears with probability $(1-o(1))p\ell/t$, using the Gale-Ryser theorem on r-factors in bipartite graphs (see, e.g. [20]), we show that with probability $1-o(1/t)$ there exists $L := (1-o(1))mp_{in}$ edge-disjoint perfect matchings $\{\mathcal{M}_{j,k}^{(i)}\}_{k=1}^{L}$. Taking the union of the edges in the matchings $\bigcup_{j=1}^{\ell-1} \mathcal{M}_{j,k}^{(i)}$ gives m directed paths, each directed from $V_1^{(i)}$ to $V_\ell^{(i)}$ and covering $\bigcup_{j=1}^{\ell} V_j^{(i)}$. Let $P_{k,1}, \dots, P_{k,m}$ denote these paths and $\mathcal{P}_k = \{P_{k,1}, \dots, P_{k,m}\}$.

Now, for each $i \in [t]$ we look at the auxiliary digraph $\widetilde{\mathcal{D}}^{(i)}$, obtained from $\mathcal{D}^{(i)}$ by shrinking every path $P_{k,j}$ into a single vertex and keeping only out-neighbours of the last and in-neighbours of the first from each path. Note that $\widetilde{\mathcal{D}}^{(i)} \sim (s+m, (1-o(1))pn^2/\alpha t s^2)$ and thus contains L edge-disjoint Hamilton cycles with probability $1-o(1/t)$. Now, every Hamilton cycle in $\widetilde{\mathcal{D}}^{(i)}$ corresponds to a Hamilton cycle in $\mathcal{D}$ and by the construction they are all edge-disjoint. All in all, w.h.p. we have $Lt = (1-o(1))np$ edge-disjoint Hamilton cycles, as desired. □

General outline of the other proofs. In the proof of Theorems 1.2, 1.4 and 1.3 we use similar methods as in the proof above. Here we give the general outline. Suppose that we have a digraph $\mathcal{G}$ and that we wish to find many Hamilton cycles (possibly edge disjoint, or covering,...). We will first break the vertex set of our digraph into a number of pieces $V_0, V_1, \dots, V_\ell$. For each $i \in [\ell-1]$ between V_i and V_{i+1} consider the bipartite digraph $\mathcal{G}_i$ consisting of edges of $\mathcal{G}$ from V_i to V_{i+1}. We will apply matching results to all of these digraphs to get many perfect matchings directed from V_i to V_{i+1}. Taking one matching from each $\mathcal{G}_i$, we obtain a collection $\mathcal{P}$ of directed paths from V_1 to V_ℓ, which cover all of $\bigcup_{i=1}^{\ell} V_i$. The aim is now to use edges incident to V_0 to complete each $\mathcal{P}$ into a Hamilton cycle.

In order to do this, for each collection of paths $\mathcal{P}$ which we wish to complete to a Hamilton cycle, we will assign it an individual sparse random subdigraph of $\mathcal{G}$ consisting of edges incident to $V_0 \cup V_1 \cup V_\ell$. Provided that $\mathcal{G}$ had certain expansion properties, this random subdigraph will also share these properties. Using these sparse random subdigraphs we will then construct auxiliary subdigraphs for each such $\mathcal{P}$ in which a directed Hamilton cycle corresponds to a directed Hamilton cycle in $\mathcal{G}$ containing the paths $\mathcal{P}$. Lastly, using the expansion properties of these auxiliary digraphs we will then guarantee Hamilton cycles, as required.

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